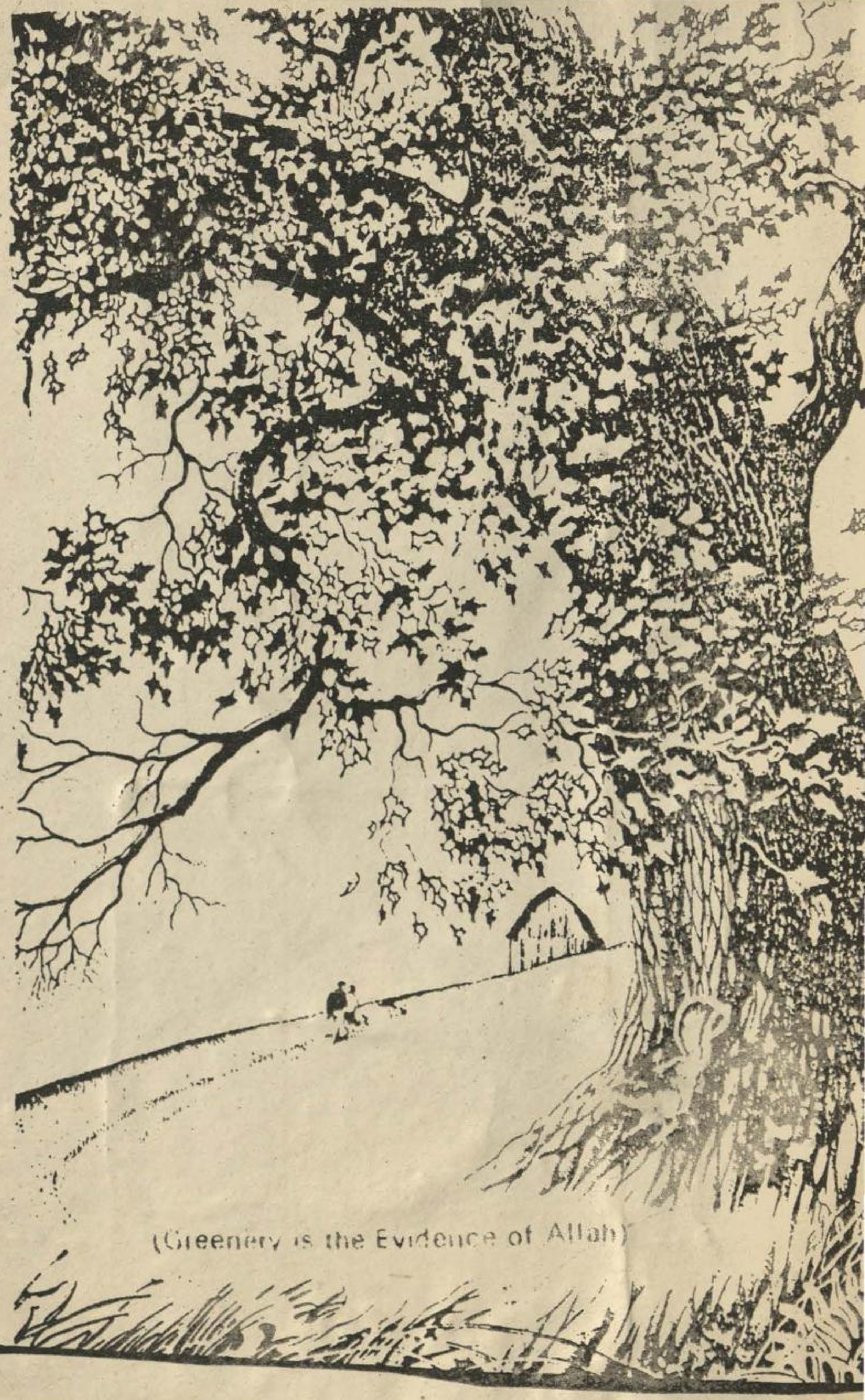
A TEXTBOOK
OF
ALGEBRA
AND
TRIGONOMETRY
CLASS XI

Published by:

PUNJAB TEXTBOOK BOARD, LAHORE



(Greenery is the Evidence of Allah)

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PREFACE

This book is meant as a textbook of Mathematics for class XI students of the Colleges/Higher Secondary Schools in the Punjab. The book has been written according to the latest syllabus prepared by the Curriculum Wing, Ministry of Education, Government of Pakistan.

The first attempt at modernisation of Intermediate Mathematics syllabus was made during the late sixties. At that time adequate and satisfactory changes in the curriculum could not be made because the students had studied only classical Mathematics at the school stage and so their background, as regards new concepts, was poor. The syllabi of all the school classes, from I to X, were, however, revised during the eighties. Therefore, it become possible, rather necessary to revise and update the syllabi of the Intermediate classes once more. This book is being published by the Punjab Text-book Board according to the modernised syllabus under its 'Open Competition Scheme'. Manuscripts were invited from authors of the whole country and those received by the Board were got evaluated by neutral experts. The units adjudged as best were combined into a single book.

All the authors and the editors have done their best to produce an ideal book. It is, however, for the readers to see how far they have succeeded in their efforts.

We are much grateful to the National Textbooks Review Committee for many useful suggestions regarding improvement of the text.

Now a few words for the teachers and the taught: It is a well-known fact that a good teacher of Mathematics is one who avoids spoon-feeding of his students - who suggests proof of a theorem or solution of a difficult problem in place of re-producing it himself on the board. The students are advised to work hard on the subject during the whole year, rather than run after keys, guides, guess-papers or other short-cuts. they should grasp the concepts thoroughly and should also memorise all the important formulae of Algebra and Trigonometry which will prove a good asset to them while studying other branches of Mathematics.

We shall be thankful to the readers for pointing out errors in the text and offering suggestions for improvement thereof.

May, 1990

EDITOR
Prof. Sh. Muhammad
Hafeez Shami

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بسم الله الرحمن الرحيم

Chapter 1

Sets and Numbers

1.1 What is Algebra?

The word 'algebra' is a variant of the Arabic word 'al-jabr' which refers to the origination of this important branch of Mathematics through the publication of the first systematic treatise bearing the title 'Ilmul-jabr-wal-muqabalah'. This book was written in the year 825 A.D. by the Muslim mathematician and astronomer Abu Abdullah Muhammad Ibn-e-Musa Al-Khawarizami. He was a native of Khawarzam and flourished at the court of the caliph Al-Mamun.

Literally 'jabr' means 'compression'. It also means 're-setting a broken bone' or 'restoring to the original position'. In choosing the title of his treatise Al-Khawarizami seems to have used the word 'jabr' in the latter sense. The phrase 'Jabra wal Maqabalah' has therefore, been , translated as 'restoration and comparison'. An example will make this meaning clear.

Example

$$\text{given } px + 2q = x^2 + px - q$$

Al–jabr gives. $px + 2q + q = x^2 + px$

Al–Muqabalah gives. $3q = x^2$.

Thus 'al-jabr' means changing negative terms into positive ones by transposition and 'al-muqabalah' means the operation of combining like or comparable terms.

Al-Khawarizmi's treatise mainly deals with solutions of algebraic equations which was originally the main function of algebra. However, in recent years much advancement has been made in the subject and now-a-days it aims at studying properties of different sets of numbers, besides, of course, solving all types of equations by employing modern techniques.

1.2 Sets, Members and Subsets

We are familiar with the notion of a 'set' because this word is frequently used in everyday speech, for instance, water set, tea set, sofa set. It is a wonder that mathematicians have developed this ordinary word into a mathematical concept in such a manner that it has become a language which is now employed in most branches of mathematics.

For the purpose of mathematics a 'set' is generally described as *"a well-defined collection of distinct objects"*. By a *"well defined collection"* is meant a collection which is such that, given any object in the world, we should be able to decide whether the object belongs to the collection or not. By *'distinct objects'* we mean 'objects no two of which are identical.'

Sometimes words 'collection', 'class' or 'family' are also used in place of the word 'set.' Mathematical examples of a set are: the set of vertices of an octagon, the set of all real numbers, the set of points of a circle, the solution set of a given equation.

Sets are generally denoted by the upper case letters A, B, C, ..., S, T, ...

The objects making up a set are called its *elements or members*. The symbol for membership is $\in$. Thus the statement 'a is a member of A' is written symbollically as

$$a \in A$$

The contrary statement 'a is not a member of A' is written as

$$a \notin A.$$

Elements or *members* of an abstract set are denoted by lower-case letters a, b, c, ...

There are three ways of describing a set.

(i) ***The Descriptive Method:*** A set may be described by means of a sentence, e.g.

The set of all the vowels of the English alphabets.

(ii) ***The Tabular Method:*** In this method a set is described by writing its elements and enclosing them with braces. Thus the above set may be written as

$$\{a, e, i, o, u\}$$

(iii) ***Set Builder Notation:*** In this notation we use a (variable) symbol, usually x, to denote any member of the set and also mention the property or properties which is / are to be possessed by x. In this notation the above set is written

$\{ x \mid x \text{ is a vowel of the English alphabet} \}$

This is read: 'The set all x such that x is a vowel of the English alphabet.'

Subsets

If every element of a set A is also an element of a set B, then A is said to be a *sub-set* of B. Symbolically this is written as

$A \subseteq B$ *(A is a sub-set of B)*

In such a case we say that B is a *superset* of A and write

$B \supseteq A$ *(B is superset of A)*

The definition of a subset may also be stated thus

$A \subseteq B$ iff $x \in A \Rightarrow x \in B$

(A is a subset of B if and only if $x \in A$ implies $x \in B$)

If A is a subset of B and B contains at least one element which is not an element of A, then A is said to be a *proper subset* of B. In such a case we write

$A \subset B$ (A is a proper subset of B)

In other words $A \subset B$ iff $A \subseteq B \wedge B \not\subseteq A$. ($\wedge$ = and)

If $A \subseteq B$ and $B \subseteq A$, then A is said to be an *improper subset* of B (simultaneously B is an improper subset of A).

In fact in such a case A and B have exactly the same elements i.e. $A = B$.

Since $A = A$ for every set A, it is reasonable to say that *every set is (an improper) subset of itself.*

Clearly, N, the set of natural numbers is a (proper) subset of Z, the set of integers and Z is a (proper) subset of R, the set of real numbers.

Combining these two statements we may say

$$N \subset Z \subset R$$

1.3 The Empty Set

Just as in the Algebra of ordinary numbers it is useful to define the number zero and have a symbol for it, in the theory of sets also it is of much advantage to define the empty set and give it a symbolic name.

Definition

A set having no element is called the *empty set* or the ***null set***. It is denoted by the symbol ϕ, or by { }.

Thus the solution set of the eq: $3x + 4 = 0$ when the domain set of x is N is the empty set.

The intersection set[*] of the line and the circle shown in the Figure 1.1 is also empty.

Note

In ordinary language a 'set' means a 'collection' and there are ordinarily at least two objects in a collection. In mathe-

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[*] Intersection set of two figures is the set containing their common points.

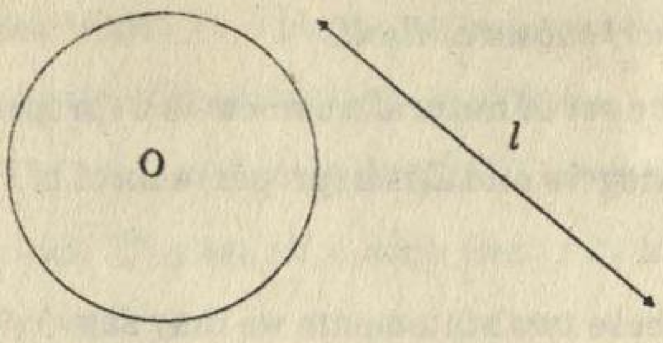

Fig. 1.1

matics we modify the meaning of the word to some extent so as to allow a set of one member only or a set having no member at all.

Remember that

The empty set is a subset of every set.

We may convince ourselves about the truth of this statement by rewording the definition of a sub-set as follows:-

"A set A is a subset of a set B if A contains no element which is not an element of B"

Given any set A, ϕ does not contain any element which is not an element of A. Therefor, ϕ is a subset of A.

Note

1. Clearly, the empty set is a proper subset of every non-empty set.
2. Since $\phi = \phi$, the empty set is an improper set of itself.

Singleton

A set having only one element is called a 'singleton.'

Fig. 1.2

The set consisting of the natural satellite of the Earth is a singleton.

1.4 Equivalent Sets, Finite sets, Infinite sets

Two sets are said to be *equivalent* if at least one (1 – 1) correspondence can be set up between their elements.

Taking a simple example, the sets { 1, 2, 3 } and { *a*, *b*, *c* } are equivalent. Two (1 – 1) correspondences between them are shown below.

$$\begin{tikzcd}[column sep=small] \{ & 1, \arrow[d, leftrightarrow] & 2, \arrow[d, leftrightarrow] & 3, \arrow[d, leftrightarrow] & \} \\ \{ & a, & b, & c, & \} \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \{ & 1, \arrow[d, leftrightarrow] & 2, \arrow[d, leftrightarrow] & 3, \arrow[d, leftrightarrow] & \} \\ \{ & a, & c, & b, & \} \end{tikzcd}$$

Can more (1 – 1) correspondences be established between them? If so, how many ?

Finite sets

(i) If a non-empty set *A* is equivalent to the set { 1, 2, 3, .., *n*} where *n* is some fixed natural number, then *A* is called, a finite set.

(ii) The empty set is regarded as a finite set.

A set which is not finite is called an *infinite set*.

Examples of finite sets are:

The set of all Pakistanis;

The set $\{1, 2, 3, \ldots, 999999\}$;

The set of all heavenly bodies existing at present.

Examples of infinite sets are:

N, the set of all natural numbers;

Z, the set of all integers;

Q, the set of all rational numbers;

R, the set of all real numbers;

The set of all rational numbers belonging to the interval $(0, 1)$;

The set of points belonging to a line-segment. 25 cm long.

Note

(1.) An interesting property of an infinite set is that it can be shown to be equivalent to *a proper subset* of itself. Thus the set of natural numbers is equivalent to the set of positive even integers:

$$\begin{array}{ccccccc} \{1, & 2, & 3, & 4, & 5, & 6, & \ldots\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \{2, & 4, & 6, & 8, & 10, & 12, & \ldots\} \end{array}$$

(2.) A finite set is not equivalent to any of its proper sub-sets.

It is evident that two finite sets are equivalent iff they have the same number of elements.

1.5 The Power Set

It is just possible that the elements of a set may themselves be sets. For example:

The set of sections of a class.

The set of classes in a college.

The set of sides of a triangle.

(A side of a triangle is itself a set of points)

An important set whose elements are sets is the power set of a set.

The *power set* of a set S is the set whose elements are all the possible subsets of S.

The power set of S is denoted by $P(S)$. Thus.

$$P(\{a,b\}) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$$

$$P(\phi) = \{\phi\}$$

Notice that the power set of the empty set is not empty. It is a singleton.

It is easy to see that if S contains n elements $P(S)$ has 2^n elements.

1.6 The Universal Set, Complements

When we are studying any branch of mathematics, the sets with which we have to deal are generally subsets of one largest set. Such a set is called the *Universal Set* or the *Universe of*

Discourse. At the elementary level when studying arithmetic the students have to deal with sets of natural numbers only. At that stage N can be regarded as the Universal set. At a later stage when the students have to deal with sets of fractions, positive and negative numbers, Q, the set of rational numbers may be considered to be the Universal set.

For clarifying certain concepts of the set theory a set of a few elements is sometimes considered to be the Universal set. Clearly this is an academic necessity.

The universal set is often denoted by the letter U.

Complement of a Set[*]

The complement of a set A, denoted by A', is the set of all elements of the universal set U which do not belong to A.

Symbolically $A' = \{x \mid x \notin A \wedge x \in U\}$

For example, if

$U = N$, the set of all natural numbers,

E = the set of all even natural numbers,

O = the set of all odd natural numbers,

then $E' = O$ and $O' = E$.

---

[*] Some authors of books on Abstract Algebra define complement of a set A relative to an arbitrary set B as the set

$\{x \mid x \in B \wedge x \notin A\}$

But we consider it preferable at this stage to follow the traditional course. Ordinarily 'complement of a set' means its complement relative to the Universal set.

Evidently,

$\varphi' = U$ and $U' = \phi$

and $(A')' = A$, for all $A \subset U$

($\because$ negation of $x \notin A$ is $x \in A$)

1.7 Operations on Sets

Just as operations of addition and subtraction are performed on two numbers, operations of union and intersection are performed on pairs of sets.

1. Union of two sets

The union of two sets A and B denoted by $A \cup B$, is defined as follows:

$A \cup B = \{ x \mid x \in A \vee x \in B \}$ $(\vee = \text{or})$

For example, if $A = \{ 1, 2, 3 \}$, $B = \{ 2, 3, 4, 5 \}$

then $A \cup B = \{ 1, 2, 3, 4, 5 \}$

Notice that the elements common to A and B, namely 2, 3, have been written only once in the expression for $A \cup B$ because repetition of an element in a set is not allowed.

2. Intersection of Two Sets

The intersection of two sets A and B, denoted by $A \cap B$, is defined by the equation:

$A \cap B = \{ x \mid x \in A \wedge x \in B \}$ $(\wedge = \text{and})$

For the sets A and B of the above example,

$A \cap B = \{ 2, 3 \}$

Disjoint sets

If $A \cap B = \phi$, A and B are called disjoint sets.

For example, if

O = The set of odd natural numbers,

E = The set of even natural numbers,

then O and E are disjoint sets.

Overlapping sets

Two sets A and B are said to be overlapping sets if $A \cap B \neq \phi$ and neither of them is a subset of the other.

Note

The following results follow from the definitions of union and intersection:-

For any set A,

(i) $A \cup U = U$ (ii) $A \cup \phi = A$,

(iii) $A \cap \mathrm{U} = A$, (iv) $A \cap \phi = \phi$.

Example

If $L = \{2, 3, 4, 5, 6\}$, $M = \{5, 6, 7, 8, 9, 10\}$

then $L \cap M = \{5, 6\} \neq \phi$

Moreover, neither L is a subset of M, nor M is a subset of L. Therefore, L and M are overlapping sets.

3. Difference Between Two Sets

If A and B are two sets then the difference between them, in that order, is denoted by $A - B$ or $A \setminus B$ (read as A slash B) and it is the set of all such elements of A as do not belong to B.

Symbolically,

$$A \setminus B = A - B = \{x \mid x \in A \wedge x \notin B\}$$

Similarly the difference between B and A, in that order, is defined as

$$B \setminus A = B - A = \{x \mid x \in B \wedge x \notin A\}$$

Example

Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{4, 5, 6, 7, 8, 9, 10\}$

Then $A \setminus B = A - B = \{1, 2, 3\}$

and $B \setminus A = B - A = \{7, 8, 9, 10\}$

Note

1. In view of the definitions of 'complement' and 'set difference' it is evident that

   $A' = U - A$

   where U is the universal set.

2. From the definition of 'complement' it follows that

   $A - B = A \cap B'$

3. Clearly, $A \cup A' = U$ and $A \cap A' = \phi$

1.8 DE-Morgan's Laws

If A and B are any sets and A', B' are their complements relative to the universal set U, then

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

These laws can be briefly stated thus:

The complement of a union is the intersection of the complements and the complement of an intersection is the union of the complements.

Proof

(i) Suppose $x \in (A \cup B)'$

Then $x \notin (A \cup B)$

$\Rightarrow x \notin A \wedge x \notin B$ (Note this argument)

$\Rightarrow x \in A' \wedge x \in B'$

$\Rightarrow x \in A' \cap B'$

Thus $x \in (A \cup B)' \Rightarrow x \in A' \cap B'$

$\therefore (A \cup B)' \subseteq A' \cap B'$ ... (1)

Suppose now $y \in A' \cap B'$

Then $y \in A' \wedge y \in B'$

$\Rightarrow y \notin A \wedge y \notin B$

$\Rightarrow y \notin A \cup B$

$\Rightarrow y \in (A \cup B)'$

Thus $y \in A' \cap B' \Rightarrow y \in (A \cup B)'$

This means $A' \cap B' \subseteq (A \cup B)'$ ... (2)

From (1) and (2), $(A \cup B)' = A' \cap B'$

Note To prove the equality of two sets we prove that either set is a subset of the other.

(ii) In law (i) replacing A by A' and B by B', we get,

$(A' \cup B')' = (A')' \cap (B')'$.

$= A \cap B$

Taking complement of each side and interchanging the two sides

$$(A\cap B)' = [(A'\cup B')']'$$

$$= A'\cup B'$$

Thus $(A\cap B)' = A'\cup B'$

1.9 Operations on Three Sets

If A, B and C are three given sets, operations of union and intersection can be preformed upon them in the following ways:

(i) $A\cup(B\cup C)$ (ii) $(A\cup B)\cup C$

(iii) $A\cap(B\cap C)$ (iv) $(A\cap B)\cap C$

(v) $A\cap(B\cup C)$ (vi) $A\cup(B\cap C)$

(vii) $(A\cup B)\cap C$ (viii) $(A\cap B)\cup C$

These combined operations satisfy certain properties or laws as described below.

1. Associative Properties

Both union and intersection are associative, that is,

(i) $A\cup(B\cup C) = (A\cup B)\cup C$

and (ii) $A\cap(B\cap C) = (A\cap B)\cap C$

The proofs of these properties are quite simple and are left as an exercise for students.

2. Distributive Properties

Union is distributive over intersection and intersection is distributive over union i.e.,

(i) $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(In algebra of real numbers multiplication is distributive over addition but addition is not distributive over multiplication.)

Proof

(i) Suppose $x \in A \cup (B \cap C)$

Then $x \in A$ or $x \in B \cap C$

Let $x \in A$. Then $x \in A \cup B \wedge x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$ ... (a)

Next suppose $x \in B \cap C \Rightarrow x \in B \wedge x \in C$

Then $x \in A \cup B \wedge x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$ ... (b)

From (a) and (b) we conclude that

$x \in A \cup (B \cap C) \Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ ... (1)

Conversely, suppose that

$y \in (A \cup B) \cap (A \cup C)$ ... (c)

There are two cases (a) $y \in A$, (b) $y \notin A$.

when $y \in A$, $y \in A \cup (B \cap C)$

When $y \notin A$, y must belong to B as well as to C [by (c)]

$\therefore y \in B \cap C$

$\Rightarrow y \in A \cup (B \cap C)$

Thus in both cases (a) and (b),

$y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in A \cup (B \cap C)$

$$\Rightarrow (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \qquad (2)$$

Hence, from (1) and (2),

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(ii) This may be proved either as above, or by complementation as follows:-

Taking complement of each side of (i) we get,

$$[A \cup (B \cap C)]' = [(A \cup B) \cap (A \cup C)]'$$

$$\Rightarrow A' \cap (B \cap C)' = (A \cup B)' \cup (A \cup C)'$$

$$\Rightarrow A' \cap (B' \cup C') = (A' \cap B') \cup (A' \cap C')$$

Replacing A', B', C' by A, B, C, we get

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example 1

Prove that $A \cap B' = A$ if $A \cap B = \phi$

Solution

(i) Suppose $A \cap B' = A$

Then $(A \cap B') \cap B = A \cap B$

$\Rightarrow$ $A \cap (B' \cap B) = A \cap B$...(Associative Property)

$\Rightarrow$ $A \cap \phi = A \cap B.$ $\qquad (B' \cap B = \phi)$

$\Rightarrow$ $\phi = A \cap B$ $\qquad (\because A \cap \phi = \phi)$

Thus $A \cap B = \phi$

(ii) Conversely, suppose that

$A \cap B = \phi$

Then $A \cap B = \phi = A \cap \phi$

$= A \cap (B' \cap B)$

$$= (A \cap B') \cap B$$

Hence $A = A \cap B'$

Example 2

Prove that $A - (B \cup C) = (A - B) \cap (A - C)$

Solution

L.H.S.

$$A - (B \cup C) = A \cap (B \cup C)'$$

$$= A \cap (B' \cap C') = A \cap B' \cap C'$$

Again, R.H.S.

$$= (A - B) \cap (A - C)$$

$$= (A \cap B') \cap (A \cap C')$$

$$= [(A \cap B') \cap A] \cap C' \qquad \text{(Associative Property)}$$

$$= [A \cap (A \cap B')] \cap C' \qquad \text{(commutative Property)}$$

$$= [(A \cap A) \cap B'] \cap C' \qquad \text{(Associative Property)}$$

$$= [A \cap B'] \cap C' \qquad (A \cap A = A)$$

$$= A \cap B' \cap C'$$

= L.H.S.

Example 3

If A, B, are any sets prove that

(i) $A - B = A - (A \cap B)$,

(ii) $(A \cap B) \cup (A - B) = A$

Solution

(i) R.H.S.

$$= A \cap (A \cap B)'$$

$= A \cap (A' \cup B')$ (De Morgan's Law)

$= (A \cap A') \cup (A \cap B')$

$= \phi \cup (A \cap B')$

$= A \cap B'$

$= A - B$

= L.H.S

(ii) L.H.S.

$= (A \cap B) \cup (A \cap B')$

$= A \cap (B \cup B')$ (Distributive Property)

$= A \cap U$

$= A$

= R.H.S

Exercise 1.1

1. Which of the following sets are finite and which sets are infinite?
   - (i) The set of students of your class.
   - (ii) The set of natural numbers between 3 and 7
   - (iii) The set of rational numbers between 3 and 7
   - (iv) The set of all stars on the sky.
   - (v) $\{1, 2, 3, 4, \ldots\}$
   - (vi) $\{1, 2, 3, 4, \ldots, 1000000000\}$
   - (vii) $\{x \mid x^2 + 16 = 0 \wedge x \in R\}$
2. Which of the following statements are true and

which of them are false?

(i) $\{1, 2, 3\} = \{3, 2, 1\}$ (ii) $\phi \subseteq \{\{0\}\}$

(iii) $\{a\} \subseteq \{\{a\}\}$ (iv) $\{a\} \in \{\{a\}\}$

3. What is the number of elements of the power set of a set having

(i) 3 elements, (ii) 5 elements,

(iii) 18 elements?

4. Prove the associative and distributive properties of union and intersection of sets.

5. Prove De-Morgan's laws.

6. Verify the distributive property of union and intersection for the sets:

$A = \{1, 4, 7, 10, 13, 16\}$, $B = \{3, 5, 7, 11, 13, 15\}$,
$C = \{3, 10, 11, 16\}$

7. Find $A - B$ and $B - A$ for the following sets

$A = \{2, 5, 7, 9, 13, 18, 19\}$,

$B = \{1, 5, 6, 8, 9, 14, 17, 18\}$

8. Describe the following sets by the tabular method

(i) $\{x \mid x^2 - 2x - 3 = 0 \wedge x \in R\}$

(ii) $\{x \mid x^2 + 4 = 0 \wedge x \in R\}$

If A, B, C, are any sets, prove the following:-

9. (i) $A \cap \phi = \phi$ (ii) $A \cup \phi = A$

(iii) $A \setminus \phi = A$ (iv) $\phi \setminus A = \phi$

10 $A \setminus (A \setminus B) = A \cap B$

11 $(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$

12. $(A \setminus B) \setminus C) = (A \setminus C) \setminus (B \setminus C)$

13. $A \subseteq B$ if

(i) $B' \subseteq A'$, (ii) $A \cap B = A$,

(iii) $A \cup B = B$, (iv) $A \setminus B = \phi$

14. $A \cap B = \phi$ iff

(i) $A \setminus B = A$, (ii) $B \setminus A = B$

15. $A = B$ iff $A \setminus B = B \setminus A$

16. $A \cup (A' \cap B) = A \cup B$

17. $(A \cap B) \cup (A' \cap B) = B$

1.10 The Real Number System

We have already an informal and intuitive familiarity with most of the properties of the real numbers. We shall now state them in a systematic axiomatic form. Before doing so we introduce a preliminary concept.

Binary Operations

Definition: A binary operation $*$ on a non-empty set A is a function from $A \times A$ into A, that is, a function whose domain is the Cartesian product $A \times A$, consisting of ordered pairs of the elements of A and whose range is a sub - set of A.

The element of A associated by $*$ with an ordered pair $(a,b) \in A \times A$, is usually denoted by $a * b$ [instead of $*(a, b)$].

The fact that $*$ is a binary operation on a set A is usually replaced by the statement "The set A is closed under the operation $*$"

For instance, addition and multiplication are binary operations on N, the set of natural numbers. For

(replacing $*$ by $+$ and $\cdot$ successively)

$$a, b \in N \Rightarrow a + b \in N$$

$$\text{and } a, b \in N \Rightarrow a . b \in N$$

The Real Number System

The real number system consists of a non-empty set R together with two binary operations addition (+) and multiplication (·) and the ordering relation. (<) (described hereinafter).

If $a, b \in R$ the element of R associated by + with the ordered pair (a, b) is denoted by $a + b$ and is called the sum of a and b.

Also the element of R associated by · with (a, b) is denoted by $a \cdot b$ and is called the product of a and b.

The operations of addition and multiplication and the ordering relation defined by (<) (less than) are supposed to satisfy certain axioms (usually called properties). Subject to these properties the elements of R are called real numbers.

I. Properties of Addition

A-1: Associative property:

For all $a, b, c \in R, a + (b + c) = (a + b) + c$

A-2: Commutative property

For all $a, b \in R, a + b = b + a$

A-3: Existence of (additive) identity:

There exists a unique element 0 of R such that for every $a \in R$

$$a + 0 = 0 + a = a$$

The element 0 is called additive identity element.

A-4: Existence of (additive) inverse:

Corresponding to every element a of R, there exists a unique element $-a$ of R such that

$$a + (-a) = (-a) + (a) = 0$$

The element $-a$ is called the additive inverse or negative of a. Also a is additive inverse of $-a$.

The difference between $a, b \in R$, in that order, is defined:

$$a - b = a + (-b).$$

The operation $-$ is called subtraction.

II. Properties of Multiplication

M-1: Associative Property:

For all $a, b, c \in R$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

M-2: Commutative Property:

For all $a, b, c \in R$, $a \cdot b = b \cdot a$

M-3: Existence of (multiplicative) identity:

There exists a unique element 1 belonging to R such that for every $a \in R$,

$$a \cdot 1 = 1 \cdot a = a$$

The element 1 is called the (multiplicative) identity element.

Def. If $a \neq 0$, a is called non-zero.

M 4: Existence of (Multiplicative) Inverse

Corresponding to every non-zero element a of R there exists a unique element $a^{-1} \in R$ such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

a^{-1} is also written as $\frac{1}{a}$

a^{-1} is called the multiplicative inverse or reciprocal of a. Clearly, a is the multiplicative inverse of a^{-1}.

The quotient of a and b is defined:

$$\frac{a}{b} = a \cdot b^{-1} = a.b^{-1} \quad [\, b \neq 0 \,]$$

$\frac{a}{b}$ is also written as $a \div b$

Thus $a \div b = a \cdot \frac{1}{b}$

Note: Multiplication sign between two elements of R, denoted by lower case letters, is generally omitted. Thus

$a \cdot b$ is written ab

$a \cdot (b \cdot c)$ is written $a(bc)$

III. M-A: The multiplication - addition Property

(The Distributive property)

For all $a, b, c \in R$,

$a(b + c) = ab + ac$ ... (Left distributivity)

and $(a + b)c = ac + bc$ ... (Right distributivity)

This property is more elaborately expressed by saying that multiplication (of real numbers) is distributive over addition.

Note: A set with two binary operations, addition and multiplication satisfying all the properties in section I, II, III above, is called a field. Thus R is a field.

Properties of Equality

Before stating the order properties, possessed by elements of R, we state the properties of the concept of equality which are often employed in various situations.

(1) Reflexive property: For every $a \in R$, $a = a$

(2) Symmetric property: For all $a, b \in R$,

$a = b \Rightarrow b = a$

(3) Transitive property: For all $a, b, c \in R$,

$a = b \wedge b = c \Rightarrow a = c$

Note: From the above properties of equality and properties of addition and multiplication stated earlier, many other properties may be deduced. e.g.,

(i) Additive property (of equality): For all $a, b, c \in R$,

$a = b \Rightarrow a + c = b + c$

$a = b$ means a and b are the same number. Therefore, in any context either of them may be replaced by the other. So $a + c$ is the same number as $b + c$

$\therefore a + c = b + c$

Similarly, we may prove the next property.

(ii) Multiplicative property: For all $a, b, c \in R$,

$a = b \Rightarrow ac = bc$

(iii) Cancellation property for addition.

For all $a, b, c \in R$,

$a + c = b + c \Rightarrow a = b$

(Hint: Using the additive property add $-c$ to both sides of the equality $a + c = b + c$)

(iv) Cancellation property for multiplication:

For all $a, b, c \in R$,

$ac = bc \Rightarrow a = b$

(v) For all $a, b, c, d \in R$,

$a = b \wedge c = d \Rightarrow a + c = b + d$

$a = b \wedge c = d \Rightarrow ac = bd$

(Other deductions are listed in the next Exercise).

IV. Properties of the Order Relation

There is an order relation among elements of R which satisfies the following properties:-

(i) Trichotomy property: For all $a, b \in R$,

either, $a = b$ or $a < b$ or $b < a$,

[$a < b$ is read 'a is less than b'. When $a < b$, then $b > a$ (b is greater than a)]*

(ii) Transitive property: For all $a, b, c \in R$,

$a < b \wedge b < c \Rightarrow a < c$

(iii) Additive property: For all $a, b, c \in R$,

$a < b \Rightarrow a + c < b + c$

(iv) Multiplicative property: For all $a, b, c \in R$,

(1) and $c > 0$ (or $0 < c$), $a < b \Rightarrow ac < bc$

(2) and $c < 0$ (or $0 > c$), $a < b \Rightarrow ac > bc$

(3) For all $a, b, c, d \in R$, and a, b, c, d all positive,

$a < b \wedge c < d \Rightarrow ac < \text{bd}$

Notes

(1) If the two sides of an inequality are interchanged, the sign of the inequality will have to be reversed. i.e. < will become > and vice versa.

(2) From the multiplicative property of inequality we conclude that

(i) if both the sides of an inequality are multiplied by a positive number, the sign of inequality remains unchanged, but

(ii) if both the sides are multiplied by a negative number, the sign of inequality is reversed.

(3) Since a and $-a$ are additive inverses or negatives of each other,

$-(-a) = a$

(Read it thus: negative of negative of a is equal to 'a'. Do not read it 'minus minus a is equal to a').

There is a delicate difference between 'negative of *a* number' and *a* 'negative number'.

'Negative of a number' means 'additive inverse of that number, whereas 'negative number' means a number belonging to the set of negative numbers*. Thus '$-a$' will be read 'negative of a' but '- 4' will be read 'negative 4'.

(4) Since a and a^{-1} are multiplicative inverses of each other

$$\left(a^{-1}\right)^{-1} = a \qquad \text{or } \frac{1}{\frac{1}{a}} = a$$

(5) Using the additive property of equality we can easily show that

$$a - b = c \qquad \text{iff} \qquad a = b + c$$

This result gives a practical method of performing subtraction.

Similarly, using the multiplicative property we can show that

$$\frac{a}{b} = c \text{ iff } a = bc$$

This results yields a practical method of performing division.

Examples

In these examples all the letters stand for real numbers.

(1) Show that $(a + b) \div c = \frac{a}{c} + \frac{b}{c}$

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* Axiom of positive and negative numbers is beyond the level of Intermediate students. So this matter is left to their intuition.

Proof

$$(a+b) \div c = \frac{a+b}{c} = (a+b) \cdot \frac{1}{c}$$

$$= a \cdot \frac{1}{c} + b \cdot \frac{1}{c}$$

$$= \frac{a}{c} + \frac{b}{c}$$

$$= a \div c + b \div c$$

Notes: Similarly, it may be shown that

$$(a-b) \div c = a \div c - b \div c$$

But it is easy to see that

$$a \div (b+c) \neq a \div b + a \div c$$

We may, therefore, so say that

'Division is distributive from the right side only. It is not distributive from the left'.

(2) Prove that (i) $a \cdot 0 = 0$ (ii) $ab = 0 \Rightarrow a = 0 \vee b = 0$

Proof

(i) For all $a \in R$

$$a \cdot 0 = a(0+0) = a \cdot 0 + a \cdot 0$$

Adding $-(a \cdot 0)$ to both sides, we have

$$-(a \cdot 0) + (a \cdot 0) = -(a \cdot 0) + a \cdot 0 + a \cdot 0$$

$$\Rightarrow 0 = 0 + a \cdot 0$$

$$\Rightarrow a \cdot 0 = 0$$

(ii) Let $ab = 0$ ... (1)

Suppose a $\neq 0$, then a^{-1} exists

(1) $\Rightarrow a^{-1}(ab) = a^{-1}\,0 = 0$ ... [By (i)]

$\Rightarrow (a^{-1}a)b = 0$ ... (Associative Property of multiplication).

$\Rightarrow 1 \cdot b = 0$

$\Rightarrow b = 0$

Similarly if $b \neq 0$, then $a = 0$

Thus $ab = 0 \Rightarrow a = 0$ or $b = 0$

(3) Prove that

(i) $(-1)a = -a$

(ii) $(-a)b = -ab$

(iii) $(-a)(-b) = ab$

Proof

(i) $(-1)a + a = (-1)a + 1 \cdot a$

$= (-1+1)a$ ... (Distributive property)

$= 0 \cdot a$

$= 0$

$\therefore$ additive inverse of a is $(-1)a$, but additive inverse of a is unique, which is $-a$

$\therefore (-1)a = -a$

(ii) $(-a)b + ab = (-a+a)b$

$= 0 \cdot b$

$= 0$

$\therefore$ additive inverse of ab is $(-a)b$, but additive inverse of ab is unique, which is $-ab$,

$\therefore (-a)b = -(ab)$

$= -ab$

(iii) $(-a)(-b) - ab = (-a)(-b) + (-ab)$ (Definition of subtraction)

$= (-a)(-b) + (-a)b$ [By (ii)]

$= (-a)(-b + b)$

$= (-a) \cdot 0$

$= 0$

$\therefore (-a)(-b) = ab \quad (x - y = 0 \Rightarrow x = y)$

(4) **Prove that**

(i) $\frac{a}{b} = \frac{c}{d}$ **if** $ad = bc$ **(Principle of equality of fractions)**

(ii) $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$

(iii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ **(Principle of Product of fractions)**

(iv) $\frac{a}{b} = \frac{ka}{kb}$ $(k \neq 0)$ **(Golden rule of fractions)**

(v) $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$

Proof

(i) $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b}(bd) = \frac{c}{d}(bd)$

$\Rightarrow a \cdot \frac{1}{b} \cdot (bd) = c \cdot \frac{1}{d}(bd)$

$$\Rightarrow a \cdot \left(\frac{1}{b} \cdot b\right) d = c \left(\frac{1}{d} bd\right)$$

$$= c\,(bd \cdot \frac{1}{d})$$

$$= cb\,(d \cdot \frac{1}{d})$$

$$= ad = cb$$

$$\text{or } ad = bc$$

Conversely,

$$ad = bc \Rightarrow ad \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right) = bc \cdot \left(\frac{1}{b} \cdot \frac{1}{d}\right)$$

$$\Rightarrow a \cdot \frac{1}{b} \cdot d \cdot \frac{1}{d} = b \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d}$$

$$\Rightarrow \frac{a}{b} = \frac{c}{d}$$

(ii) $$(ab) \cdot \frac{1}{a} \cdot \frac{1}{b} = (a \cdot \frac{1}{a})\,(b \cdot \frac{1}{b}) = 1 \cdot 1 = 1$$

Thus $a\,b$ and $\frac{1}{a} \cdot \frac{1}{b}$ are multiplicative inverses of each other.

But multiplicative inverse of ab is $\frac{1}{ab}$.

$$\therefore \frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$$

(iii) $$\frac{a}{b} \cdot \frac{c}{d} = (a \cdot \frac{1}{b})\,(c \cdot \frac{1}{d})$$

$$= ac \cdot \frac{1}{b} \cdot \frac{1}{d} = ac \cdot \frac{1}{bd}$$

$$= \frac{ac}{bd}$$

(iv) $$\frac{a}{b}=\frac{a}{b}\cdot 1=\frac{a}{b}\left(k\cdot\frac{1}{k}\right) \qquad (k\neq 0)$$

$$=\frac{a}{b}\cdot\frac{k}{k}=\frac{ak}{bk} \qquad [\text{ by (iii)}]$$

$$\therefore \quad \frac{a}{b}=\frac{ka}{kb}$$

(v) $$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{\frac{a}{b}(bd)}{\frac{c}{d}(bd)} \quad \ldots [\text{ by (iv)}]$$

$$=\frac{ad\cdot\frac{1}{b}\cdot b}{cb\cdot\frac{1}{d}\cdot d}=\frac{ad}{bc}$$

$$\therefore \quad =\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{ad}{bc}$$

1.11 The Set of Rational Numbers

The set of real numbers has some important sub-sets:

N = Set of natural numbers.

Z = Set of integers (positive integers, zero, and negative integers).

Q = Set of rational numbers.

Q' = Set of irrational numbers.

We know that a rational number is such a real number as can be written in the form p/q where p and q are integers but

$q \neq 0$ *

An irrational number is a number which cannot be written in the farm of p/q.

It is interesting to note that the set of rational numbers possesses all the properties of R, the set of real numbers. The difference between Q and R is that R satisfies completeness axiom (to be described in higher classes) whereas Q does not.

We know that square root of an imperfect square is an irrational number, e.g. $\sqrt{3}$, $\sqrt{5}$, $\sqrt{11}$ are all irrational numbers. Similarly, cube root of an imperfect cube is irrational, e.g. $\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{4}, \sqrt[3]{25}$.

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* It is interesting to examine why division of a non-zero number by zero is disallowed in mathematics. We know that

$$\frac{a}{b} = x \text{ iff } bx = a \quad \dots (1)$$

Four cases arise for consideration:-

(i) If $a \neq 0, b \neq 0$, eq (1) has a unique solution:-

So $\frac{a}{b}$ has a unique value.

(ii) If $a = 0, b \neq 0$, eq. (1) $\Rightarrow bx = 0 \Rightarrow x = 0$

$$\therefore \frac{a}{b} = 0 \text{ i.e } \frac{0}{b} = 0$$

(iii) If $a \neq 0, b = 0$, eq. (1) $0x = a \Rightarrow 0 = a$

Thus $0 = a$ non-zero number.

Therefore, division of a non-zero number by zero leads to a contradiction. Hence it is not allowed.

(iv) If $a = 0, b = 0$, eq (1) $\Rightarrow 0x = 0$ which is satisfied for all real values of x.

Hence $\frac{0}{0}$ is arbitrary. It can have any value.

Clearly (iii) and (iv) represent extreme situations.

A very interesting irrational number is π. Its approximate value is $\frac{22}{7}$, a better approximation is $\frac{355}{113}$ and still a better approximated value is 3.141592654. Since it is an irrational number its exact value in the form $\frac{p}{q}$ or in the form of a recurring or terminating decimal fraction cannot be determined.

Example

Prove that $\sqrt{2}$ is irrational

Proof

Suppose, if possible, $\sqrt{2} = \frac{p}{q}$ ... (1)

where $q \neq 0$, and p, q are in lowest terms (for otherwise they can be reduced to such terms before starting).

Squaring both sides of (1), we get

$$2q^2 = p^2 \quad \ldots \quad (2)$$

In this equation, L.H.S. is even. Therefore, so should be R.H.S. Since the square of an even number only can be even, p must be even.

Let $p = 2p'$. Then (2) becomes

$$2q^2 = 4p'^2$$

$$\Rightarrow q^2 = 2p'^2$$

Arguing as before q must be even.

Thus p and q are both even and, therefore, have a common factor 2.

This contradicts the hypothesis that p and q are in lowest terms.

Since the supposition (1) has led to a contradiction, it is impossible.

$\therefore \sqrt{2}$ is irrational.

Exercise 1.2

1. Which of the following sets have closure property w.r.t. addition and multiplication?

(i) {0} (ii) {1} (iii) {0,– 1}, (iv) {1 ,– 1}

2. Name the properties used in the following equations. Letters, where used, represent real numbers:-

(i) $4 + 9 = 9 + 4$

(ii) $(a + 1) + \frac{3}{4} = a + (1 + \frac{3}{4})$

(iii) $(\sqrt{3} + \sqrt{5}) + \sqrt{7} = \sqrt{3} + (\sqrt{5} + \sqrt{7})$

(iv) $100 + 0 = 100$

(v) $1000 \times 1 = 1000$

(vi) $4.1 + (-4.1) = 0$

(vii) $a - a = 0$

(viii) $\sqrt{2} \times \sqrt{5} = \sqrt{5} \times \sqrt{2}$

(ix) $a\,(b - c) = ab - ac$

(x) $(x - y)\,z = xz - yz$

(xi) $4 \times (5 \times 8) = (4 \times 5) \times 8$

(xii) $a(b + c - d) = ab + ac - ad$

3. Name the properties used in the following inequalities

(i) $-3 < -2 \Rightarrow 0 < 1$

(ii) $-5 < -4 \Rightarrow 20 > 16$

(iii) $1 > -1 \Rightarrow -3 > -5$

(iv) $a < 0 \Rightarrow -a > 0$

(v) $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$

(vi) $a > b \Rightarrow -a < -b$

4. Prove the following rules of addition

(i) $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

(ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

5. Prove that $\frac{7}{12} - \frac{5}{18} = \frac{21-10}{36}$

6. Simplify, justifying each step:-

(i) $\frac{4+16x}{4}$ (ii) $\frac{\frac{1}{4}+\frac{1}{5}}{\frac{1}{4}-\frac{1}{5}}$

(iii) $\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}$ (iv) $\frac{\frac{1}{a}-\frac{1}{b}}{1-\frac{1}{a}\cdot\frac{1}{b}}$

Prove the following

7. The additive inverse of a number is unique.

8. The multiplicative identity is unique.

9. $-(x+y) = -x-y$

10. If $b \neq 0, d \neq 0,$ $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

11. $-\frac{-x}{y}=-\frac{x}{-y}=\frac{-x}{-y}=\frac{x}{y}$

12. $0 < x < y \Leftrightarrow 0 < \frac{1}{y} < \frac{1}{x}$

13. $(a < b) \wedge (c < d) \Rightarrow a+c < b+d$

14. If a and b are non-negative, then

$a^2 < b^2 \Leftrightarrow a < b$

1.12 The Complex Number System

The complex number system is the set $C = R \times R$ whose elements (ordered pairs of real numbers) satisfy the following properties. Subject to these properties elements of C are called complex numbers.

P-1: $(a, b) = (c, d)$ if $a = c \wedge b = d$

P-2: $(a, b) + (c, d) = (a+c, b+d)$

P-3: If k is any real number, then $k(a, b) = (ka, kb)$

P-4: $(a, b)(c, d) = (ac-bd, ad+bc)$

P-1, P-2, P-4 define respectively equality, sum and product of two complex numbers.

P-3 defines the product of a real number and a complex number.

Theorem

The set C satisfying the above four properties is a field.

Proof

First we verify the properties of addition.

A-0: Addition as defined above is a binary operation on C.

If $(a, b), (c, d) \in C$,

$(a, b) + (c, d) = (a + c, b + d) \in C$

$\therefore$ addition is a binary operation.

A-1: Let $(a, b), (c, d), (e, f) \in C$. Then

$(a, b) + [(c, d) + (e, f)] = (a, b) + (c + e, d + f)$

$= (a + c + e, b + d + f)$

$= [(a + c) + e, (b + d) + f]$

$[(a, b) + (c, d)] + (e, f)$

$\therefore$ addition is associative.

A-2: It is trivially simple to verify that

$(a, b) + (c, d) = (c, d) + (a, b)$

$\therefore$ addition in C is commutative.

A-3: The identity element for addition in C is $(0,0)$

for $(a, b) + (0, 0) = (0, 0) + (a, b) = (a, b)$

A-4: The additive inverse of every element of C exists, for

$(a, b) + (-a, -b) = (0, 0)$

$\therefore (-a, -b)$ is the additive inverse of (a, b)

II Properties of Multiplication

M-0: Product of two elements of C given by P - 4 is another element of C.

∴ multiplication in C is a binary operation.

M-1: Multiplication is associative.

Verify yourselves.

M-2: Multiplication is commutative.

Verification is quite simple.

M-3: Multiplicative identity is $(1, 0)$

$$\begin{aligned} \text{For } (a, b)(1, 0) &= (a \cdot 1 - b.0, b \cdot 1 + a \cdot 0) \\ &= (a, b) \\ &= (1, 0)(a, b) \text{ (as can be easily verified)} \end{aligned}$$

M-4: Multiplicative inverse of every non-zero element of C [element ≠ 0] exists.

In fact inverse of (a, b) is $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$ *

for $(a, b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1, 0)$

III Multiplication - Addition Property

It can be easily verified that

$$(a, b)[(c, d) + (e, f)] = (a, b)(c, d) + (a, b)(e, f).$$

---

* Inverse of (a, b) can be found as follows:-

Let $(x, y) \in C \quad (a, b)(x, y) = (1, 0)$

$\Rightarrow (ax - by, ay + bx) = (1, 0)$

$\Rightarrow ax - by = 1, ay + bx = 0$

Solving these we get,

$$x = \frac{a}{a^2+b^2}, \; y = \frac{-b}{a^2+b^2}$$

Thus C satisfies all the field axioms.

1.13 A Special Sub-set of C

We consider a sub-set of C whose elements are of the form $(a, 0)$ i.e second component of each element of which is zero.

Let $(a, 0)$, $(c, 0)$ be the two elements of the subset. Then

(i) $(a, 0) + (c, 0) = (a + c, 0)$

(ii) $k(a, 0) = (ka, 0)$

(iii) $(a, 0)(c, 0) = (ac, 0)$

(iv) Multiplicative inverse of $(a, 0) = \left(\frac{1}{a}, 0\right)$ (as can be easily calculated).

Notice that these results are same as we should have obtained if we had operated on the real numbers 'a' and 'c' ignoring the second number of each ordered pair i.e. 0 which has played no part in the above calculations.

On account of this special feature we identify the Complex number $(a, 0)$ with the real number 'a' i.e.,

We postulate

$$(a, 0) = a \quad \ldots \quad (1)$$

Now consider $(0, 1)$.

$$(0, 1) \cdot (0, 1) = (-1, 0)$$

$$= -1 \qquad \text{[by (1) above]}$$

If we set $(0, 1) = i \quad \ldots \quad (2)$

then $(0, 1)^2 = i^2 = -1$

We are now in a position to write every complex number, given as an ordered pair, in terms of i. For

$$\begin{aligned}(a, b) &= (a, 0) + (0, b) \quad \text{(Definition. of addition)}\\ &= a(1, 0) + b(0, 1)\\ &= a \cdot 1 + bi \qquad \text{[By (1) and (2) above]}\\ &= a + bi\end{aligned}$$

Thus $(a, b) = a + bi$ where $i^2 = -1$.

This result enables us to convert any complex number given in one notation into the other.

Exercise 1.3

1. Derive the addition properties A-1 to A-4 of complex numbers.
2. Derive the multiplicative properties M-1 to M-4 of complex numbers.
3. Derive the distributive law of complex numbers.

$$(a, b)\,[(c, d) + (e, f)] = (a, b)\,(c, d) + (a, b)\,(e, f)$$

4. Simplify:

(i) i^9, (ii) i^{14}, (iii) $(-i)^{-19}$

5. Write in terms of i:

(i) $\sqrt{-1}\,b$ (ii) $\sqrt{-5}$ (iii) $\sqrt{\frac{-16}{25}}$

Simplify the following:-

6. $(7, 9) + (3, -5)$ 7. $(8, -5) - (-7, 4)$

8. $(2, 6)\,(3, 7)$ 9. $(5, -4)\,(-3, -2)$

10. $(0, 3)\,(0, 5)$ 11. $(2, 6) \div (3, 7)$

[Hint for 11: $\dfrac{(2, 6)}{(3, 7)} = \dfrac{(2+6i)}{(3+7i)} \times \dfrac{(3-7i)}{(3-7i)}$ etc.]

12. $(6, 4) \div (3, 8)$

13. Prove that the sum as well as the product of any two numbers of the form $a + ib$, $a - ib$ is a real number

14. Find the multiplicative inverse of each of the following numbers.

(i) $(-4, 7)$ (ii) $(\sqrt{2}, -\sqrt{5})$

15. Factorise

(i) $a^2 + 4b^2$ (ii) $9a^2 + 16b^2$ (iii) $2x^2 + 3y^2$

16. Seperate into real and imaginary parts (write as a single complex number)

(i) $\dfrac{2-7i}{4+5i}$ (ii) $\left(\dfrac{-2+3i}{1+i}\right)^2$

1.14 Conjugate of a Complex Number.

If $z = x + iy$, $x, y \in R$, then the complex number $x - iy$ is called the conjugate of z and is usually denoted by $\bar{z}$.

In such a case z is also called conjugate of $\bar{z}$. Thus $\bar{\bar{z}}$ is the same number as z.

Notice that conjugate of a complex number is obtained by changing the sign of the imaginary term.

Thus $\overline{-i} = i$, $\overline{1-i} = 1+i$,

$\overline{-4+2i} = -4-2i$, $\overline{5i-6} = -5i-6$

Theorem

If z_1, z_2 are complex numbers, then

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(iii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

Proof

Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

(i) Then $z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$

$$\begin{aligned}\therefore\ \overline{z_1 + z_2} &= (x_1 + x_2) - i\,(y_1 + y_2)\\ &= x_1 - iy_1 + x_2 - iy_2\\ &= \overline{z_1} + \overline{z_2}\end{aligned}$$

(ii)
$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i\,(x_1 y_2 + x_2 y_1)$$

$$\overline{z_1 z_2} = x_1 x_2 - y_1 y_2 - i\,(x_1 y_2 + x_2 y_1) \quad \text{... (1)}$$

Also $\overline{z_1}\,\overline{z_2} = (x_1 - iy_1)\,(x_2 - iy_2)$

$$= (x_1 x_2 - y_1 y_2) - i\,(x_1 y_2 + x_2 y_1) \quad \text{... (2)}$$

Hence $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(iii) Do yourselves (First simplify $\dfrac{z_1}{z_2}$ and then take the conjugate)

1.15 Geometrical Representation of Complex Numbers

The Complex Plane

We have seen that there is a (1 – 1) correspondence between the elements (ordered pairs) of the Cartesian product $R \times R$ and the points of the coordinate plane. Also there is a (1 – 1) correspondence between the elements of $R \times R$ and the complex numbers. Therefore, there is a (1 – 1) correspondence between the points of the coordinate plane and the complex numbers. We can, therefore, represent complex numbers by points of the coordinate plane. In this representation every complex number will be represented by one and only one point of the coordinate plane and every point of the plane will represent one and only one complex number. The components of the complex number will be the coordinates of the point representing it. In this representation the x-axis is called the real axis and the y-axis is called the imaginary axis. The coordinate plane itself is called the Complex plane. By way of illustration a number of complex numbers have been shown in figure 1.3.

The figure representing one or more complex numbers on the complex plane is called an Argand diagram.

Points on the x-axis represent real numbers whereas the points on the y-axis represent imaginary numbers.

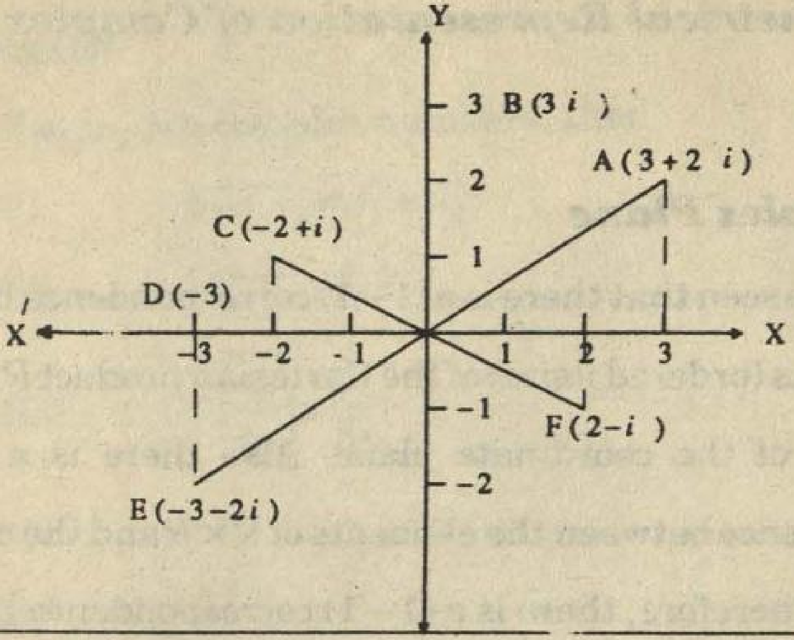
Y
3 B(3i)
A(3+2i)
2
C(−2+i)
1
D(−3)
X
X
−3
−2
−1
1
2
3
−1
F(2−i)
−2
E(−3−2i)

Fig. 1.3

Modulus of a Complex Number

In fig 1.4 (a, b) are the coordinates of a point A. It represents the complex number $a + ib$.

The number $\sqrt{a^2 + b^2}$ is called the modulus of the complex number $a + ib$.

In the fig. 1.4 $\overline{MA} \perp \overrightarrow{OX}$

$\therefore OM = a, MA = b$

In the right-angled traingle OMA, we have, by Pythagoraos

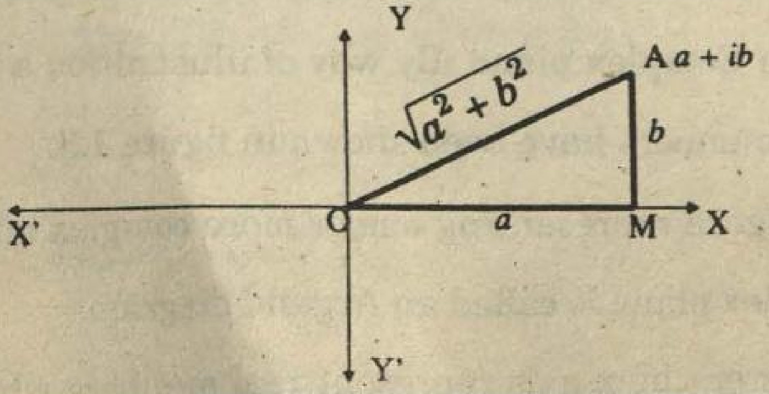
Y
A a + ib
$\sqrt{a^2+b^2}$
b
X'
O
a
M
X
Y'

Fig. 1.4

Theorem,

$$OA^2 = OM^2 + MA^2$$

$$\therefore\ OA = \sqrt{a^2 + b^2}$$

Thus OA represents the modulus of $a + ib$. In other words:

The modulus of a complex number is the distance from the origin of the point representing the number.

The modulus of a complex number is generally denoted as

$$|a + bi| \text{ or } |(a, b)|$$

For convenience, complex numbers are named z_1, z_2, etc.

If $z = a + ib = (a, b)$, then

$$|z| = \sqrt{a^2 + b^2}$$

Theorem

$\forall\ z_1, z_2 \in C$

(i) $|z_1 z_2| = |z_1|\ |z_2|$.

(ii) $|z_1| - |z_2| < |z_1 + z_2| \leq |z_1| + |z_2|$

Proof

Let $z_1 = a + bi$, $\quad z_2 = c + di$

$$\begin{aligned} \text{Then } |z_1 z_2| &= |(a + ib)(c + id)| \\ &= |(ac - bd) + (ad + bc)\,i| \\ &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} \\ &= \sqrt{a^2c^2 + a^2d^2 + b^2d^2 + b^2c^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \end{aligned}$$

$$= \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2}$$

$$= |z_1| \cdot |z_2|$$

This result may be stated thus

The modulus of the product of two complex numbers is equal to the product of their moduli.

(ii) Algebraic proof of this part is tedious. Therefore, we prove it geometrically.

In the figure 1.5 point A represents $z_1 = a + bi$ and point C represents $z_2 = c + di$. We complete the $||^m$ OABC. From the figure it is evident that coordinates of B are $(a + c, b + d)$. B represents $z_1 + z_2 = (a + c) + (b + d)i$ and OB $= |z_1 + z_2|$

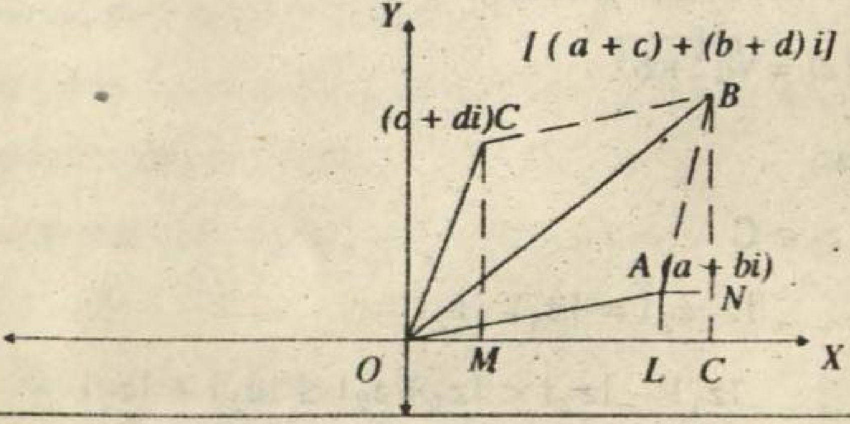

Fig. 1.5

Also OA $= |z_1|$, AB = OC $= |z_2|$

In the Δ OAB, OA + AB > OB.*

---

* In any triangle the sum of the lenghts of any two sides is greater then the length of the third side and the difference of the lengths of any two sides is less then the length of the third side.

$$\therefore \left|z_1\right| + \left|z_2\right| > \left|z_1 + z_2\right| \quad \ldots \quad (1)$$

Also in the same triangle, OA – AB < OB

$$\therefore \left|z_1\right| - \left|z_2\right| < \left|z_1 + z_2\right| \quad \ldots \quad (2)$$

Combining (1) and (2), we have

$$|z_1| - |z_2| < |z_1 + z_2| < |z_1| + |z_2| \quad \ldots (3)$$

which gives the required results with inequality signs.

Results with equality sign will hold when the points A and C representing z_1 and z_2 become collinear with B.

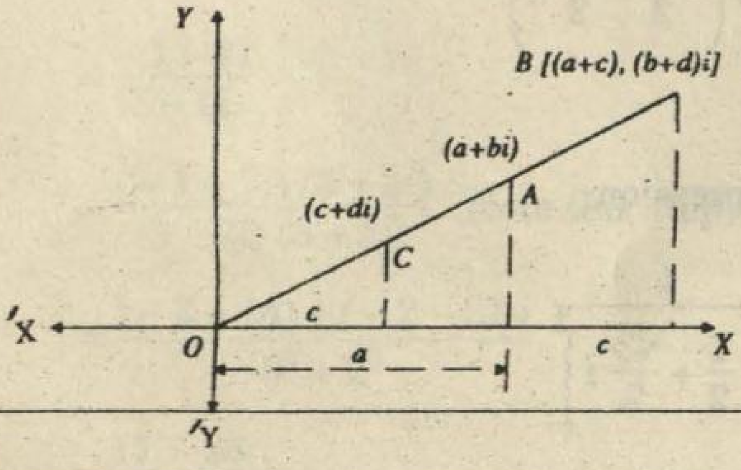

Fig. 1.6

This will be so when $\frac{a}{b} = \frac{c}{d}$. (See fig. 1.6 above)

In such a case

$$\begin{aligned} |z_1| + |z_2| &= OA + OC \\ &= OA + AB \\ &= OB \quad (AB = OC) \\ &= |z_1 + z_2| \end{aligned}$$

Thus $|z_1 + z_2| = |z_1| + |z_2|$

N.B. the second part of result (ii), namely,

$$|z_1 + z_2| \le |z_1| + |z_2|$$

is called the triangular inequality. In words, it may be stated thus:

The modulus of the sum of two complex numbers is less than or equal to the sum of the moduli of the two numbers.

Miscellaneous Examples

Example 1.

Evaluate, $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{-3}$

Solution

Given expression

$$=\frac{1}{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3}$$

$$=\frac{1}{\left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2\cdot\frac{\sqrt{3}}{2}i+3\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)^2+\left(\frac{\sqrt{3}}{2}i\right)^3}$$

$$=\frac{1}{-\frac{1}{8}+3\frac{\sqrt{3}}{8}i-\frac{9}{8}i^2+3\frac{\sqrt{3}}{8}i^3}$$

$$=\frac{8}{-1+3\sqrt{3}\,i+9-3\sqrt{3}\,i}$$

$$=\frac{8}{8}=1$$

Example 2.

If $z_1 = 2 + i$, $z_2 = 3 - 2i$, $z_3 = 1 + 3i$,

express $\dfrac{\overline{z_1}\,\overline{z_3}}{z_2}$ in the form $a + bi$.

Solution

$$\frac{\overline{z_1}\,\overline{z_3}}{z_2} = \frac{\overline{(2+i)}\,\overline{(1+3i)}}{(3-2i)}$$

$$= \frac{(2-i)(1-3i)}{(3-2i)}$$

$$= \frac{(2-3)+(-6-1)i}{(3-2i)}$$

$$= \frac{-1-7i}{3-2i}$$

$$= \frac{(-1-7i)(3+2i)}{(3-2i)(3+2i)} \quad \text{(Note this step)}$$

$$= \frac{(-3+14)+(-2-21)i}{3^2+2^2}$$

$$= \frac{11}{13} - \frac{23}{13}i$$

Example 3.

Show that $\forall\, z_1, z_2 \in \mathrm{C}, \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

Solution

Let $z_1 = a + bi$, $\qquad z_2 = c + di$

$$\overline{z_1 z_2} = \overline{(a+bi)(c+di)}$$

$$= \overline{(ac-bd)+(ad+bc)i}$$

$$= (ac-bd)-(ad+bc)i \qquad (1)$$

Also $\overline{z_1}\,\overline{z_2} = \overline{(a+bi)}\,\overline{(c+di)}$

$$= (a-bi)(c-di)$$

$$= (ac-bd) + (-ad-bc)i$$

$$= (ac-bd) - (ad+bc)i \qquad \ldots \qquad (2)$$

(1) and (2) give the required result.

Exercise 1.4

1. Graph the following numbers on the complex plane

(i) $2+3i$ (ii) $2-3i$

(iii) $-2-3i$ (iv) $-2+3i$

(v) -6 (vi) $4i$

(vii) $\frac{3}{5}-\frac{4}{5}i$ (viii) $-5-6i$

2. Find the multiplicative inverses of the following numbers

(i) $-3i$ (ii) $1-2i$ (iii) $-3-5i$

3. Simplify

(i) i^{101} (ii) $(-ai)^4$, a being a real number

(iii) i^{-3} (iv) i^{-10}

4. Prove that $\bar{z} = z$ if z is real.

5. Simplify and express in the form $a+bi$

(i) $(5+2\sqrt{-4})(3-\sqrt{-4})$

(ii) $(2+\sqrt{-3})(3-\sqrt{-3})$

(iii) $\dfrac{2}{\sqrt{5}+\sqrt{-8}}$ (iv) $\dfrac{3}{\sqrt{6}-\sqrt{-12}}$

6. Show that $\forall\, z \in C, z^2 + \bar{z}^2$ is a real number.

7. Criticize: "$z^2 - \bar{z}^2$ is a real number."

8. Simplify:

(i) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^3$

(ii) $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{-3}$

(iii) $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^{-2} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$

(iv) $(a + bi)^2$ (v) $(a + bi)^{-2}$ $(a \neq 0, b \neq 0)$

(vi) $(a + bi)^3$

(vii) $(a - bi)^3$

(viii) $(3 - \sqrt{-4})^{-3}$

9. Show that $\left|\frac{1 + 2i}{2 - i}\right| = 1$

10. Find the conjugate of

(i) $-6 - i$ (ii) $4\,i - 3$ (iii) $(-3, 1)$

11. Prove that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$,

12. If $z_1 = 3 - 2\,i$, $z_2 = 1 + i$, *evaluate*

(i) $|4z_1 - 5z_2|$, (ii) $\left(\overline{z_1}\right)^2$

Chapter 2

Matrices and Determinants.

2.1 Introduction

We are already familiar with the concept of matrices upto order 2 × 2, their addition and multiplication etc. We now consider matrices of higher orders over real numbers. If a matrix A has m rows and n columns, we say A is a *matrix of order* $m \times n$. If $m \neq n, A$ is called *rectangular matrix*. If $m = n$, A is called *square matrix*. If we use double subscript notation for the elements of a matrix A, so that a_{ij} denotes the element in the ith row and jth column of A, then A can be written in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdot & \vdots & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

For the sake of brevity, A can also be written as $A = [a_{ij}]$, $i = 1,2,\ldots,m, j = 1,2,\ldots,n$, when we say $A=[a_{ij}]$ is $m \times n$, we omit

$i = 1,2,\ldots, m, j = 1,2,\ldots, n$.

Examples

1. $[a,b]$ is 1×2 matrix.

2. $\begin{pmatrix} a \\ b \end{pmatrix}$ is 2×1 matrix

3. $\begin{pmatrix} -1 & 2 & 6 \\ 3 & 0 & -2 \end{pmatrix}$ is 2×3 matrix

4. $\begin{pmatrix} -1 & 3 & 5 \\ 2 & 0 & 4 \\ 6 & -2 & 1 \end{pmatrix}$ is 3×3 matrix.

5. $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & 5 \\ -1 & 0 & 4 \\ 3 & 1 & 2 \end{pmatrix}$ is 4×3 matrix.

2.2 Algebra of Matrices

1. Equality

Two matrices A and B are equal, $A = B$, if and only if

(i) A and B are of the same order and

(ii) their corresponding elements are equal.

Thus if $A=[a_{ij}]$ and $B = [b_{ij}]$, are $m \times n$ matrices, and $a_{ij} = b_{ij}$ for all i and j, then $A = B$.

Examples

1.
$$\begin{pmatrix} 5+4 & 3-7 & 2-1 \\ 2+8 & 6+1 & 3+4 \\ 5-7 & 2-1 & 6-2 \end{pmatrix} = \begin{pmatrix} 9 & -4 & 1 \\ 10 & 7 & 7 \\ -2 & 1 & 4 \end{pmatrix}$$

2.
$$\begin{pmatrix} 3\times 2 & 4\times 5 \\ 5\times 2 & 7\times 5 \\ 4\times 3 & 2\times 4 \end{pmatrix} = \begin{pmatrix} 6 & 20 \\ 10 & 35 \\ 12 & 8 \end{pmatrix}$$

3. $$\begin{pmatrix} 2 & 1 & 5 & 4 \\ 3 & 0 & 7 & 8 \\ 4 & 8 & 6 & 9 \end{pmatrix} = \begin{pmatrix} \frac{4}{2} & 2-1 & \sqrt{25} & \sqrt{16} \\ \frac{9}{3} & 2-2 & \sqrt{49} & \sqrt{64} \\ 2^2 & 2^3 & 2\times 3 & 3^2 \end{pmatrix}$$

Note that

$$\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 & 5 \\ 3 & 0 & 4 \end{pmatrix}$$

Also

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{pmatrix}$$

2. Addition of Matrices

Two matrices having the same order are said to be *conformable for addition*.

The sum of two matrices of the same order is found by adding the corresponding elements of the two matrices.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then $A + B = [a_{ij} + b_{ij}]$ is also $m \times n$ matrix.

Example 1

$$\text{Let } A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 4 \end{pmatrix}$$

$$\text{and } B = \begin{pmatrix} 3 & 2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$$

$$\text{Then } A + B = \begin{pmatrix} 1+3 & 0+2 & -1+1 \\ 0-2 & -2+1 & 4-2 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & 2 & 0 \\ -2 & -1 & 2 \end{pmatrix}$$

Addition of matrices of the same order is both commutative and associative.

Let A, B, C be $m \times n$ matrices

Suppose $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$.

Now $A + B = [a_{ij}] + [b_{ij}]$

$= [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ ($\because$ addition is commutative in the set of real numbers)

$= [b_{ij}] + [a_{ij}]$

$= B + A$

Also $(A + B) + C$

$= ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}]$

$= [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij} + (bij + c_{ij})]$ ($\because$ + is associative in real numbers)

$= [a_{ij}] + [b_{ij} + c_{ij}]$

$= [a_{ij}] + ([b_{ij}] + [c_{ij}])$

$= A + (B + C)$

Thus addition of matrices of the same order is both commutative and associative.

Example 2

$$\text{Let} \quad A=\begin{pmatrix}1 & 0 & -1\\ 2 & 3 & 4\\ 0 & 1 & 2\end{pmatrix}$$

$$B=\begin{pmatrix}2 & 1 & 4\\ 3 & 2 & 0\\ 1 & -1 & 2\end{pmatrix}$$

$$\text{and} \quad C=\begin{pmatrix}1 & 3 & 4\\ 2 & 3 & 5\\ 0 & 3 & 4\end{pmatrix}$$

$$\text{Here} \quad A+B=\begin{pmatrix}1+2 & 0+1 & -1+4\\ 2+3 & 3+2 & 4+0\\ 0+1 & 1-1 & 2+2\end{pmatrix}$$

$$=\begin{pmatrix}3 & 1 & 3\\ 5 & 5 & 4\\ 1 & 0 & 4\end{pmatrix}$$

$$B+A=\begin{pmatrix}2+1 & 1+0 & 4-1\\ 3+2 & 2+3 & 0+4\\ 1+0 & -1+1 & 2+2\end{pmatrix}$$

$$=\begin{pmatrix}3 & 1 & 3\\ 5 & 5 & 4\\ 1 & 0 & 4\end{pmatrix}$$

Thus $A + B = B + A$

Similarly $A + C = C + A$ and $B + C = C + B$ (Verify)

$$\text{Now} \quad (A+B)+C=\begin{pmatrix}3 & 1 & 3\\ 5 & 5 & 4\\ 1 & 0 & 4\end{pmatrix}+\begin{pmatrix}1 & 3 & 4\\ 2 & 3 & 5\\ 0 & 3 & 4\end{pmatrix}$$

$$=\begin{pmatrix}3+1 & 1+3 & 3+4\\ 5+2 & 5+3 & 4+5\\ 1+0 & 0+3 & 4+4\end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 7 \\ 7 & 8 & 9 \\ 1 & 3 & 8 \end{pmatrix}$$

$$A + (B + C) = A + \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 5 \\ 0 & 3 & 4 \end{pmatrix}$$

$$= A + \begin{pmatrix} 2+1 & 1+3 & 4+4 \\ 3+2 & 2+3 & 0+5 \\ 1+0 & -1+3 & 2+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 4 & 8 \\ 5 & 5 & 5 \\ 1 & 2 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1+3 & 0+4 & -1+8 \\ 2+5 & 3+5 & 4+5 \\ 0+1 & 1+2 & 2+6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 7 \\ 7 & 8 & 9 \\ 1 & 3 & 8 \end{pmatrix}$$

Thus $(A + B) + C = A + (B + C)$

3. *The Null or Zero Matrix*

A null or zero matrix O is a matrix in which all elements are zero. An $m \times n$ null matrix is denoted by $O_{m\times n}$. Thus

$$O_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$O_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$O_{3\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ etc.}$$

When order of a null matrix is understood, we simply write

O for the null matrix.

If A is any matrix and O a null matrix of the same order as that of A, then

$$A + O = O + A = A$$

Example

$$\text{Let } A = \begin{pmatrix} 2 & 3 & 0 & 3 \\ 1 & 5 & 4 & 2 \\ -6 & 2 & 7 & 8 \end{pmatrix}$$

$$\text{and } O = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \text{Then}$$

$$A + O = \begin{pmatrix} 2 & 3 & 0 & 3 \\ 1 & 5 & 4 & 2 \\ -6 & 2 & 7 & 8 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0 & 3+0 & 0+0 & 3+0 \\ 1+0 & 5+0 & 4+0 & 2+0 \\ -6+0 & 2+0 & 7+0 & 8+0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 0 & 3 \\ 1 & 5 & 4 & 2 \\ -6 & 2 & 7 & 8 \end{pmatrix} = A$$

Similarly $O + A = A$

4. Additive Inverse of a Matrix

Let $A = [a_{ij}]$. The matrix each of whose elements is additive inverse of the corresponding element of A is denoted by $-A$. That is, $-A = [-a_{ij}]$.

$$\text{Now } A + (-A) = [a_{ij}] + [-a_{ij}]$$

$$= [a_{ij} + (-a_{ij})] = [0_{ij}] = \mathrm{O}$$

Similarly $(-A) + A = \mathrm{O}$

Thus $-A = [-a_{ij}]$ is additive inverse of A.

It is clear from the above definition that A and $-A$ are of the same order.

Example

$$\text{Let } B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\text{Then } -B = \begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{pmatrix}$$

$$B + (-B) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1 & 2-2 & 3-3 \\ 4-4 & 5-5 & 6-6 \\ 7-7 & 8-8 & 9-9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

Similarly $(-B) + B = O$. Thus $-B$ is the additive inverse of B.

5. Subtraction of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same order, then we define

$$\begin{aligned} A - B &= A + [-B] \\ &= [a_{ij}] + [-b_{ij}] \\ &= [a_{ij} - b_{ij}] \end{aligned}$$

Example

$$\text{Let } A = \begin{pmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 2 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 4 & -3 \\ 4 & -2 & 6 \\ 8 & 3 & 0 \end{pmatrix}$$

$$\text{Then } -B = \begin{pmatrix} -2 & -4 & +3 \\ -4 & +2 & -6 \\ -8 & -3 & 0 \end{pmatrix}$$

$$\text{and } A - B = A + (-B)$$

$$= \begin{pmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 2 & 1 \end{pmatrix} + \begin{pmatrix} -2 & -4 & 3 \\ -4 & 2 & -6 \\ -8 & -3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2 & 4-4 & 5+3 \\ 6-4 & 7+2 & 8-6 \\ 9-8 & 2-3 & 1+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 8 \\ 2 & 9 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

6. Scalar Multiplication

If $A = [a_{ij}]$ is a matrix of order $m \times n$ and k is any real number, then kA is a matrix obtained by multiplying each element of A by k. That is,

$kA = [ka_{ij}]$

This process of obtaining kA is called scalar multiplication of the matrix A.

It follows that order of kA is equal to order of A.

Example

$$\text{Let } A=\begin{pmatrix}2 & 3 & -4\\ 0 & 1 & 0\\ 3 & -5 & 6\end{pmatrix}$$

$$\text{Then } 2A=\begin{pmatrix}2\times 2 & 2\times 3 & 2\times(-4)\\ 2\times 0 & 2\times 1 & 2\times 0\\ 2\times 3 & 2\times(-5) & 2\times 6\end{pmatrix}$$

$$=\begin{pmatrix}4 & 6 & -8\\ 0 & 2 & 0\\ 6 & -10 & 12\end{pmatrix}$$

$$3A=\begin{pmatrix}3\times 2 & 3\times 3 & 3\times(-4)\\ 3\times 0 & 3\times 1 & 3\times 0\\ 3\times 3 & 3\times(-5) & 3\times 6\end{pmatrix}$$

$$=\begin{pmatrix}6 & 9 & -12\\ 0 & 3 & 0\\ 9 & -15 & 18\end{pmatrix}$$

$$\frac{1}{2}A=\begin{pmatrix}\frac{2}{2} & \frac{3}{2} & -\frac{4}{2}\\ \frac{0}{2} & \frac{1}{2} & \frac{0}{2}\\ \frac{3}{2} & -\frac{5}{2} & \frac{6}{2}\end{pmatrix}$$

$$=\begin{pmatrix}1 & \frac{3}{2} & -2\\ 0 & \frac{1}{2} & 0\\ \frac{3}{2} & -\frac{5}{2} & 3\end{pmatrix}$$

Note that $A+A=2A$

$A+A+A=3A$ etc.

In general, if n is a positive integer

$$\underbrace{A + A + A + \dots + A}_{n \ terms} = nA$$

Definition

A matrix of order $1 \times n$ has only one row. Such a matrix i called a *row vector*. Similarly a matrix of order $m \times 1$ has onl one column and is called a *column vector*. Thus $[0 \ 5 \ -2 \ \ 1]$ is a row vector and $\begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$ is a column vector.

7. *Multiplication of Matrices*

Two matrices A and B are said to be *conformable fo multiplication* AB if the number of columns of A is equal to th number of rows of B.

Let C be $1 \times n$ row vector and D be $n \times 1$ column vector say

$$C = [\, a_1 \ a_2 \ a_3 \ldots\ldots a_n \,] \text{ and } D = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

Then $CD = [a_1 b_1 + a_2 b_2 + \dots + a_n b_n] = a_1 b_1 + a_2 b_2$ $+ \dots + a_n b_n$ *

Suppose A is $m \times n$ matrix and B is $n \times p$ matrix.

Then the product AB is a $m \times p$ matrix C whose

---

* We agree to write 1×1 matrix $[a]$ as a

(i,j)th element.

$$C_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \ldots\ldots & a_{in} \end{bmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \cdot \\ \cdot \\ \cdot \\ b_{nj} \end{pmatrix}$$

$$= a_{i1}\, b_{ij} + a_{i2}\, b_{2j} + \ldots + a_{in}\, b_{nj}$$

$$= \sum_{\lambda=1}^{n} a_{i\lambda}\, b_{\lambda j}$$

Example 1

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Here A is 2×3 matrix and B is 3×3 matrix. Hence A and B are conformable for multiplication.

$$\text{Now } AB = \begin{pmatrix} 1(1)+2(4)+3(7) & 1(2)+2(5)+3(8) & 1(3)+2(6)+3(9) \\ 4(1)+5(4)+6(7) & 4(2)+5(5)+6(8) & 4(3)+5(6)+6(9) \end{pmatrix}$$

$$= \begin{pmatrix} 1+8+21 & 2+10+24 & 3+12+27 \\ 4+20+42 & 8+25+48 & 12+30+54 \end{pmatrix}$$

$$= \begin{pmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \end{pmatrix}$$

Example 2

$$\text{Let } A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\text{Then } A\,B = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1(2)+2(1) & 1(1)+2(1) \\ -1(2)+3(1) & -1(1)+3(1) \end{pmatrix}$$

$$= \begin{pmatrix} 2+2 & 1+2 \\ -2+3 & -1+3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2(1)+1(-1) & 2(2)+1(3) \\ 1(1)+1(-1) & 1(2)+1(3) \end{pmatrix}$$

$$= \begin{pmatrix} 2-1 & 4+3 \\ 1-1 & 2+3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 7 \\ 0 & 5 \end{pmatrix}$$

Thus $AB \neq BA$

In general, $AB \neq BA$. It may happen that AB is defined, BA is not defined.

If A, B, C are three matrices such that each of the following product is defined:

AB, BC, $A\,(BC)$, $(AB)\,C$

then $A\,(BC) = (AB)\,C$

That is, matrix multiplication is associative. If A is a square matrix, then the product AA is defined. This product is denoted by A^2. We define $A^3 = (A^2)\,A$ and so on.

Thus $A^n = (A^{n-1})\,A$

Example 3

$$\text{Let } A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & 1 & 0 \\ 3 & -5 & 6 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 2 & 3 & -4 \\ 0 & 1 & 0 \\ 3 & -5 & 6 \end{pmatrix}\begin{pmatrix} 2 & 3 & -4 \\ 0 & 1 & 0 \\ 3 & -5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2(2)+3(0)-4(3) & 2(3)+3(1)-4(-5) & 2(-4)+3(0)-4(6) \\ 0(2)+1(0)+0(3) & 0(3)+1(1)+0(-5) & 0(-4)+1(0)+0(6) \\ 3(2)-5(0)+6(3) & 3(3)-5(1)+6(-5) & 3(-4)-5(0)+6(6) \end{pmatrix}$$

$$= \begin{pmatrix} 4-12 & 6+3+20 & -8-24 \\ 0 & 1 & 0 \\ 6+18 & 9-5-30 & -12+36 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 29 & -32 \\ 0 & 1 & 0 \\ 24 & -26 & 24 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} -8 & 29 & -32 \\ 0 & 1 & 0 \\ 24 & -26 & 24 \end{pmatrix}\begin{pmatrix} 2 & 3 & -4 \\ 0 & 1 & 0 \\ 3 & -5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} -8(2)-32(3) & -8(3)+29(1)-32(-5) & -8(-4)+29(0)-32(6) \\ 0(2)+1(0)+0(3) & 0(3)+1(1)+0(-5) & 0(-4)+1(0)+0(6) \\ 24(2)-26(0)+24(3) & 24(3)-26(1)+24(-5) & 24(-4)-26(0)+24(6) \end{pmatrix}$$

$$= \begin{pmatrix} -16-96 & -24+29+160 & 32-192 \\ 0 & 1 & 0 \\ 48+72 & 72-26-120 & -96+144 \end{pmatrix}$$

$$= \begin{pmatrix} -112 & 165 & -160 \\ 0 & 1 & 0 \\ 120 & -74 & 48 \end{pmatrix}$$

Definitions

A square matrix $A = [a_{ij}]$ is called a *diagonal matrix* if $a_{ij}=0$ for all $i \neq j$ and $a_{ij} \neq 0$ when $i = j$ i,e.$a_{ii} \neq$ O. If A is $n \times n$, then the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are said to constitute the *leading diagonal* of the matrix A.

If A is a diagonal matrix and $a_{11} = a_{22} = \ldots = a_{nn} = d$, then A is called a *scalar matrix* of order n.

Furthermore if $d = 1$, then A is called the *identity* matrix of order n. It is also called the *unit matrix* of order n and is denoted by I_n.

Thus

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ etc.}$$

When the order of the identity matrix is understood, it is also denoted by I.

It can be verified that if A is $n \times n$ matrix,

then $AI_n = I_n A = A$

Example

$$\text{Let } A = \begin{pmatrix} 6 & 7 & 8 \\ 7 & 9 & 10 \\ -8 & 10 & 11 \end{pmatrix}$$

$$\text{Then } I_3 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 7 & 8 \\ 7 & 9 & 10 \\ -8 & 10 & 11 \end{pmatrix}$$

$$=\begin{pmatrix} 6 & 7 & 8 \\ 7 & 9 & 10 \\ -8 & 10 & 11 \end{pmatrix}$$

Similarly $A\,I_3 = A$

Definition

If $A = [a_{ij}]$ is $m \times n$ matrix, then the $n \times m$ matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. It is denoted by A^t.

Example

$$\text{If } A = \begin{pmatrix} 3 & 4 & 1 & 4 \\ 1 & 5 & 4 & 2 \\ -6 & 2 & 7 & 8 \end{pmatrix}$$

$$A^t = \begin{pmatrix} 3 & 1 & -6 \\ 4 & 5 & 2 \\ 1 & 4 & 7 \\ 4 & 2 & 8 \end{pmatrix}$$

Definition

A square matrix A is called *symmetric* if $A = A^t$

Example 1

$$\text{Let } A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

$$\text{Then } A^t = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

$$= A$$

Thus A is symmetric.

Example 2

$$\text{Let } A = \begin{pmatrix} 6 & 8 & -9 & 5 \\ 8 & 3 & -3 & 2 \\ -9 & -3 & 2 & 1 \\ 5 & 2 & 1 & 0 \end{pmatrix}$$

$$\text{then } A^t = \begin{pmatrix} 6 & 8 & -9 & 5 \\ 8 & 3 & -3 & 2 \\ -9 & -3 & 2 & 1 \\ 5 & 2 & 1 & 0 \end{pmatrix}$$

$$= A$$

Hence A is symmetric.

Exercise 2.1

1. Write each sum as a single matrix.

(i) $\begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \\ -1 & 5 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 4 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} 4 & -1 & 3 \\ 2 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 3 & -1 & 0 \\ 4 & 0 & -2 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & 0 & -1 \\ 2 & 1 & 2 \\ 4 & 5 & -4 \end{pmatrix} + \begin{pmatrix} 6 & -1 & 0 \\ 0 & 2 & 4 \\ -1 & -2 & -3 \end{pmatrix}$

(iv) $\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ -1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 1 \\ -3 & 0 & -2 \\ 4 & -1 & 3 \end{pmatrix}$

2. Write each product as a single matrix.

(i) $3\begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 0 \end{pmatrix}$ (ii) $-5\begin{pmatrix} 0 & 1 & -1 \\ 3 & -1 & 2 \end{pmatrix}$

(iii) $\begin{bmatrix} 3 & -2 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

(iv) $\begin{bmatrix} 2 & -2 & -1 \end{bmatrix}\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$

(v) $\begin{pmatrix} -3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 0 \end{pmatrix}$

(vi) $\begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}$

(vii) $\begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(viii) $\begin{pmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{pmatrix}$

3. Find a matrix X satisfying each matrix equation.

(i) $3X+\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & -1 & 5 \end{pmatrix}=\begin{pmatrix} -2 & 3 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 5 \end{pmatrix}$

(ii) $2X+3\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}=\begin{pmatrix} 7 & -1 & 1 \\ 3 & -5 & 1 \end{pmatrix}$

(iii) $X+2I=\begin{pmatrix} 3 & -1 & 0 \\ 1 & 2 & 3 \\ 4 & 2 & 6 \end{pmatrix}$

4. Show that if $A=\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, then

(i) $(A+B)(A+B) \neq A^2 + 2AB + B^2$

(ii) $(A+B)(A-B) \neq A^2 - B^2$

5. Let $A = \begin{pmatrix} 4 & 5 & -3 \\ 2 & 6 & 1 \\ 0 & 7 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 5 & 7 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{pmatrix}$

verify the following:

(i) $(A+B)+C = A+(B+C)$

(ii) $A+(-A) = (-A)+A = 0$

(iii) $A+B = B+A$

(iv) If c, d are any real numbers, then

(a) $c(dA) = (cd)A$,

(b) $(c+d)A = cA + dA$

(c) $c(A+B) = cA + cB$

(v) $(AB)C = A(BC)$

(vi) $A(B+C) = AB + AC$

(vii) $(B+C)A = BA + CA$

6. Verify $(AB)^t = B^t A^t$, when

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$$

7. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, show that

$$A^2 - 4A - 5I = 0$$

8. Which of the matrices are symmetric?

(i) $\begin{pmatrix} 3 & 4 & 6 \\ 4 & 0 & 2 \\ 6 & 2 & 8 \end{pmatrix}$

(ii) $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & d \end{pmatrix}$

(iii) $\begin{pmatrix} a & h & g & l \\ h & l & f & m \\ g & f & c & n \\ l & m & n & o \end{pmatrix}$

(iv) $\begin{pmatrix} u_1 & u_2 & u_3 \\ u_2 & u_4 & u_5 \\ u_3 & -u_5 & u_6 \end{pmatrix}$

9. Fill in the blanks.

(i) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$ is a __________ matrix.

(ii) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is a __________ matrix.

(iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is __________ matrix.

(iv) $\begin{pmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{pmatrix}$ is a __________ matrix.

(v) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is the __________ matrix.

2.3 Determinants and their Properties

Associated with each square matrix A over real numbers is a real number called the ***determinant*** of A. If A is a square matrix of order $n \times n$, the associated determinant is called a ***determinant of order n***. The notation $|A|$ is used to denote the determinant of A. If

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \qquad \text{we write}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We define $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$.

We define $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11} (a_{22} a_{33} - {}^{a}{}_{23} a_{32}) - a_{12} (a_{21} a_{33} - a_{23} a_{31}) +$

$a_{13} (a_{21} a_{32} - a_{22} a_{31})$

$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} +$

$a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$ ... (i)

This is known as *expansion* of $|A|$ *with the help of first row.*

We can also define

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) +$$

$$a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}$$

$$a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}$$

$$a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \ldots \text{(ii)}$$

This is called expansion of $|A|$ with the help of first column.

We find that (i) and (ii) are equal.

Thus we can use first row or first column to find the value of $|A|$.

Minor of an Element of a Matrix

Given an $n \times n$ matrix $A = [a_{ij}]$, if we delete the i th row and j th column of A, we get an $(n-1) \times (n-1)$ matrix. The determinant of this $(n-1) \times (n-1)$ matrix is called *minor of the element* a_{ij} and is denoted by M_{ij}.

Example

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Minor of a_{11}, i.e $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of a_{23} i.e $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

Minor of a_{31}, i.e $M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

Cofactors of Elements of a Matrix

Let $A = [a_{ij}]$ be a square matrix.

The Cofactor of a_{ij}, denoted by A_{ij}, is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Thus A_{ij} is a signed minor. If $i + j$ is an even integer, the '+' sign is attached to M_{ij} and if $i + j$ is odd, the negative sign is attached to M_{ij}. In other words, the Cofactor A_{ij} of the element a_{ij} is the minor M_{ij} of a_{ij} if $i + j$ is an even integer, and the additive inverse of the minor M_{ij} of a_{ij} if $i + j$ is an odd integer.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then the Cofactor A_{11} of a_{11} is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

because 1 + 1 = 2, an even integer.

The Cofactor A_{23} of a_{23} is $-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

because 2 + 3 = 5, an odd integer.

The value of $|A|$ can now be written in terms of Cofactors as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

Thus, the determinant, $|A|$, is equal to the sum formed by multiplying each entry in the first row by its Cofactors.

Similar methods can be used to show that the determinant $|A|$, is also equal to the sum formed by multiplying each entry in any row (or column) by its Cofactors. Thus we also have

$$|A| = a_{12} A_{12} + a_{22} A_{22} + a_{32} A_{32} \text{ (verify)}$$

Example 1

$$\text{Let } A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{pmatrix}, \text{ then}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 0 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 3(10) + 0 + (-5)$$

$$= 25$$

Find the expansion of $|A|$ by using elements of second (third) row. Do we get the same result? Also verify the result by expanding the determinant with the help of first (second, third) column.

Properties of Determinants

We now state some properties which will help us to evaluate determinants.

1. *If each element in any row or each element in any column of a square matrix A is 0, then* $|A| = 0$

Suppose each element of the second row in

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ is zero. Then}$$

$$a_{21} = a_{22} = a_{23} = 0$$

Now $|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$$= 0\,A_{21} + 0\,A_{22} + 0\,A_{23} = 0$$

We have similar result when each element of any other row or each element of any column is zero.

2. *If any two rows (or columns) of a square matrix are interchanged, the determinant of the resulting matrix is the additive inverse of the determinant of the original matrix.*

For example

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{21}\left(a_{12}a_{33} - a_{13}a_{32}\right) - a_{22}\left(a_{11}a_{33} - a_{13}a_{31}\right) + a_{23}\left(a_{11}a_{32} - a_{12}a_{31}\right)$$

$$= a_{21}a_{12}a_{33} - a_{21}a_{13}a_{32} - a_{22}a_{11}a_{33} + a_{22}a_{13}a_{31} + a_{23}a_{11}a_{32} - a_{23}a_{12}a_{31}$$

$$=-\left(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}\right)$$

$$=-|A|$$

3. *If two rows (or two columns) in a square matrix are identical (i.e. corresponding elements are equal), the value of the determinant is zero.*

$$\text{Let } A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{pmatrix}$$

Let A' denote the matrix obtained by interchanging th first and second rows then $A'=A$. Hence $|A'|=|A|$. B property (2) $|A'|=-|A|$. Hence $|A|=-|A|$. i.e $2|A|=$ i.e $|A|=0$. This can also be verified directly by expanding $|A$ by the first row.

Similar result follows when any two rows or any tw columns are identical.

Since $a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23}$

$$=-\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{11} & a_{12} & a_{13}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$= 0$ (by property (3))

This result can be restated more generally as follows:

The sum of the products of the elements of a row (or colum by the Cofactors of the elements of different row (or column) zero.

4. *If each of the elements of one row (or column) of a squa*

matrix is multiplied by k, then the determinant of the resulting matrix is k times the determinant of the original matrix. For example,

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= k\, a_{11} A_{11} + k\, a_{12} A_{12} + k\, a_{13} A_{13}$$

$$= k\, (a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13})$$

$$= k\, |A|$$

We have similar result when any other row or column is multiplied by k.

5. If each element in a row (or column) of a determinant *is written as the sum of two terms, then the determinant can be written as the sum of two determinants as follows:*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \alpha_{21} & a_{22} + \alpha_{22} & a_{23} + \alpha_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding the given determinant with the help of second row, we get

$$(a_{21} + \alpha_{21})\, A_{21} + (a_{22} + \alpha_{22})\, A_{22} + (a_{23} + \alpha_{23})\, A_{23}$$

$$= (a_{21} A_{21} + a_{22} A_{22} + a_{23} A_{23}) + (\alpha_{21} A_{21} + \alpha_{22} A_{22} + \alpha_{23} A_{23})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

6. *If each element of one row (or column) of a determinant is multiplied by a real number k and the resulting product is added to the corresponding element in another row (or column) respectively in the determinant, then the resulting determinant is equal to the original determinant.*

For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} + k\,a_{12} & a_{12} & a_{13} \\ a_{21} + k\,a_{22} & a_{22} & a_{23} \\ a_{31} + k\,a_{32} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{R.H.S} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k\,(0)$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The properties 1 – 6 listed above can be used to write sequences of equal determinants, leading from one form of determinant to another and more useful form.

Example 2

Evaluate the determinant of

$$A = \begin{pmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix}$$

Interchanging the first and second rows,

$$|A| = -\begin{vmatrix} 1 & 1 & 6 & 3 \\ 4 & 2 & 5 & 10 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix}$$

Multiply the first row by 4 and 7 and subtract the corresponding results from the second and third rows respectively. We have

$$|A| = -\begin{vmatrix} 1 & 1 & 6 & 3 \\ 0 & -2 & -19 & -2 \\ 0 & -4 & -42 & -16 \\ 0 & 2 & 5 & 8 \end{vmatrix}$$

$$= -\begin{vmatrix} -2 & -19 & -2 \\ -4 & -42 & -16 \\ 2 & 5 & 8 \end{vmatrix}$$

(Using elements of the first column and the corresponding cofactors).

Taking (– 1) common from both the first and second rows, we get

$$|A| = (-1)^3 \begin{vmatrix} 2 & 19 & 2 \\ 4 & 42 & 16 \\ 2 & 5 & 8 \end{vmatrix}$$

Taking the factor 2 from the first column, we have

$$|A| = -2 \begin{vmatrix} 1 & 19 & 2 \\ 2 & 42 & 16 \\ 1 & 5 & 8 \end{vmatrix}$$

Multiply the first row by 2 and 1 and subtract the corresponding results from the second and third rows respectively we get

$$|A| = -2 \begin{vmatrix} 1 & 19 & 2 \\ 0 & 4 & 12 \\ 0 & -14 & 6 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 4 & 12 \\ -14 & 6 \end{vmatrix}$$

(Using elements of the first column and their corresponding cofactors).

$$= -2\,(24 + 168)$$

$$= -2\,(192)$$

$$= -384$$

Example 3

Prove the identity using properties of determinants only.

$$\begin{vmatrix} \beta\gamma & \alpha & \alpha^2 \\ \gamma\alpha & \beta & \beta^2 \\ \alpha\beta & \gamma & \gamma^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix}$$

Denote the determinant on the left hand side by Δ. Multiply the first, second and third rows by α, β and γ respectively. Then

$$\Delta = \frac{1}{\alpha\beta\gamma}\begin{vmatrix} \alpha\beta\gamma & \alpha^2 & \alpha^3 \\ \alpha\beta\gamma & \beta^2 & \beta^3 \\ \alpha\beta\gamma & \gamma^2 & \gamma^3 \end{vmatrix}$$

$$= \frac{\alpha\beta\gamma}{\alpha\beta\gamma}\begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \alpha^2 & \alpha^3 \\ 1 & \beta^2 & \beta^3 \\ 1 & \gamma^2 & \gamma^3 \end{vmatrix}.$$

Example 4

Verify that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\text{Let } \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Subtracting the first row from second and third rows respectively, we have

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}.$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b - a)(c - a)(c + a - b - a)$$

$$= (a - b)(b - c)(c - a)$$

2.4 Singular and Non-Singular Matrices

A square matrix A is called *singular* if $|A| = 0$ and it is called *non-singular* if $|A| \neq 0$

Example 1

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Then $|A| = -2$ (Verify)

Thus A is non-singular.

Example 2

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Then $|A| = 0$ (Verify)

Hence A is a singular matrix.

2.5 Adjoint of a Matrix

Let A be a square matrix of order $n \times n$. Let $\mathcal{A}$ denote the matrix obtained by replacing each element of A by its corresponding Cofactor. Then $\mathcal{A}^t$ is called the adjoint of A and is written as adj A.

For the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$adj.\, A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Example

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$. Then

$$\mathcal{A} = \begin{pmatrix} 1 & -2 & -3 \\ -1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$$

and $adj\, A = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 0 & 2 \\ -3 & 1 & 1 \end{pmatrix}$

2.6 Inverse of a Square Matrix

For a given square matrix of order $n \times n$, if there exists a square matrix B of order $n \times n$ such that

$$AB = BA = I_n,$$

then B is called an *inverse* of A.

It may be noted that inverse of a square matrix, if it exists, is unique.

Suppose a square matrix A has two inverses B and C. Then

$$AB = BA = I_n$$

$$AC = CA = I_n$$

So $\quad B\,(AC) = B\, I_n = B$

$(BA)\,C = IC = C$

Since $B\,(AC) = (BA)\,C$ by associative law for multiplication, we have $B = C$.

Thus inverse of A is unique and we can denote the inverse of A, if it exists, by A^{-1}. Now we describe a method of finding inverse of a non-singular matrix.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Since A is non-singular, $|A| \neq 0$. Now

$$A \cdot \frac{adjA}{|A|} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \frac{A_{31}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \frac{A_{32}}{|A|} \\ \frac{A_{13}}{|A|} & \frac{A_{23}}{|A|} & \frac{A_{33}}{|A|} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}}{|A|} & \frac{a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}}{|A|} & \frac{a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}}{|A|} \\ \frac{a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}}{|A|} & \frac{a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}}{|A|} & \frac{a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33}}{|A|} \\ \frac{a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13}}{|A|} & \frac{a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23}}{|A|} & \frac{a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}}{|A|} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{|A|}{|A|} & 0 & 0 \\ 0 & \frac{|A|}{|A|} & 0 \\ 0 & 0 & \frac{|A|}{|A|} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Similarly $\frac{adj\,A}{|A|} A = I_3$

Thus $A\left(\frac{adj\,A}{|A|}\right) = \left(\frac{adj.\,A}{|A|}\right)A = I_3$

By uniqueness of the inverse of a matrix, it follows that

$$A^{-1} = \frac{adj\,A}{|A|}, \text{ when } |A| \neq 0$$

Example

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

Then $|A| = -2$, which is non zero (Prove it).

adj $A = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 0 & 2 \\ -3 & 1 & 1 \end{pmatrix}$ (See example section 2.5)

Hence $A^{-1} = \frac{adj\,A}{|A|}$

$$= -\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -2 & 0 & 2 \\ -3 & 1 & 1 \end{pmatrix}$$

$$
= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -1 \\ \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}
$$

2.7 *Homogeneous and Non-Homogeneous Linear Equations*

The linear system

$$
\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array}\right\} \quad \dots (1)
$$

Can be written as the matrix equation.

$$
\begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots\dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \dots (2)
$$

$$
\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots\dots & a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},
$$

then (2) can be written in the form

$$AX = B$$

If $B \neq 0$, then (1) is called *non-homogeneous system of linear equations*. If $B = 0$, then (1) is called a system *of homogeneous linear equations*.

Let $B \neq 0$ and A be non-singular. Then $|A| \neq 0$ and so A^{-1} exists. Multiplying both sides of $AX = B$ on the left by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$

$$\left(A^{-1}A\right)X = A^{-1}B$$

$$IX = A^{-1}B$$

$$\text{i.e} \qquad X = A^{-1}B$$

Where $A^{-1}B$ is $n \times 1$ column matrix. By equating the corresponding elements, we get the solution set of the linear system.

If A is a singular matrix, $|A| = 0$. Hence A^{-1} does not exist. In this case either the system has no solution or the solution is not unique.

Example

Use matrices to find the solution set of

$$2x + y + z = 1$$

$$x - 2y - 3z = 1$$

$$3x + 2y + 4z = 5$$

$$\text{Let} \qquad A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & 4 \end{pmatrix}$$

$$\text{Then} \quad |A| = -9$$

$$\text{and} \qquad A^{-1} = -\frac{1}{9}\begin{pmatrix} -2 & -2 & -1 \\ -13 & 5 & 7 \\ 8 & -1 & -5 \end{pmatrix}$$

$$\text{Now} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -2 & -2 & -1 \\ -13 & 5 & 7 \\ 8 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

$$= -\frac{1}{9}\begin{pmatrix} -9 \\ 27 \\ -18 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

Thus $x = 1, y = -\ 3$ and $z = 2$

and solution set is $\{1, -3, 2\}$

2.8 Cramer's Rule

Consider the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

In matrix notation, the system can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

or $AX = B$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

If $|A| = 0$, then

$$X = A^{-1}B = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\text{i.e}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} b_1 A_{11} + b_2 A_{21} + b_3 A_{31} \\ b_1 A_{12} + b_2 A_{22} + b_3 A_{32} \\ b_1 A_{13} + b_2 A_{23} + b_3 A_{33} \end{pmatrix}$$

$$\text{Hence } x_1 = \frac{b_1 A_{11} + b_2 A_{21} + b_3 A_{31}}{|A|}$$

$$= \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

$$x_2 = \frac{b_1 A_{12} + b_2 A_{22} + b_3 A_{32}}{|A|}$$

$$= \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

$$\text{similarly } x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

This relationship expresses Cramer's Rule. If $|A| \neq 0$, we replace the *j*th column of the co-efficient matrix *A* with the corresponding entry from the column matrix *B* (the constant terms in the equation), and the value of $x_j, j = 1, 2, 3$ is then given as above.

Example

Use Cramer's rule to solve the system

$$-4x + 2y - 9z = 2$$

$$3x + 4y + z = 5$$

$$x - 3y + 2z = 8$$

$$\text{Here } |A| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 63$$

$$x = \frac{\begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}}{63} = \frac{441}{63} = 7$$

$$y = \frac{\begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix}}{63} = \frac{-189}{63} = -3$$

$$z = \frac{\begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix}}{63} = \frac{-252}{63} = -4$$

Exercise 2.2

1. Evaluate each determinant

(i) $\begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & 3 & 1 \\ -1 & 2 & 1 \\ 0 & 2 & 0 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 2 & 0 & 2 \end{vmatrix}$

(iv) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{vmatrix}$

(v) $\begin{vmatrix} a & b & 1 \\ a & b & 1 \\ 1 & 1 & 1 \end{vmatrix}$

(vi) $\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$

(vii) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

2. Without evaluating, state why each statement is true.

(i) $\begin{vmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ -1 & 2 & 0 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 3 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0$

(iii) $$\begin{vmatrix} 2 & 3 & 1 & 1 \\ 2 & 0 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{vmatrix} = 0$$

(iv) $$\begin{vmatrix} 4 & 2 & 1 \\ 0 & -1 & -2 \\ 1 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 4 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix}$$

(v) $$\begin{vmatrix} 3 & -4 & 2 \\ 1 & -2 & 0 \\ 0 & 8 & 1 \end{vmatrix} = -2\begin{vmatrix} 3 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & -4 & 1 \end{vmatrix}$$

(vi) $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 3 \\ 16 & 5 & 6 \\ 25 & 8 & 9 \end{vmatrix}$$

(vii) $$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -5 & 0 \end{vmatrix}$$

(viii) $$\begin{vmatrix} -1 & 1 & 0 \\ 2 & 3 & -1 \\ 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 5 & 3 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

3. Evaluate

(i) $$\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ -1 & 2 & 0 \end{vmatrix}$$

(ii) $$\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 0 & 1 & 0 \end{vmatrix}$$

(iii) $$\begin{vmatrix} 1 & 0 & 3 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

(iv) $$\begin{vmatrix} 11 & -10 & 32 \\ 12 & -11 & 35 \\ 12 & -6 & 33 \end{vmatrix}$$

(v) $$\begin{vmatrix} 26 & 29 & 27 \\ 25 & 27 & 30 \\ 25 & 26 & 28 \end{vmatrix}$$

(vi) $$\begin{vmatrix} 4 & 2 & 0 \\ -1 & 0 & 2 \\ 3 & 0 & -1 \end{vmatrix}$$

4. Verify that

(i) $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

(ii) $$\begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix} = 0$$

(iii) $$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

(iv) $$\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$$

(v) $$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

(vi) $$\begin{vmatrix} bc & ca & ab \\ a^3 & b^3 & c^3 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix} = 0, a \neq 0, b \neq 0, c \neq 0$$

(vii) $$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

5. If A and B are non-singular matrices, show that

(i) $\left(A^{-1}\right)^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

6. Find the inverse of each matrix, if it exists.

(i) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 4 & 2 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 4 & 2 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(vi) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

7. Which of the following matrices are singular and which of them are non-singular?

(i) $\begin{bmatrix} 1 & -1 & 6 \\ 2 & 4 & -2 \\ 3 & 5 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 5 \\ \frac{1}{2} & 4 & 6 \\ 2 & 0 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 0 & 9 \\ 8 & -112 & -8 \\ 2 & 2 & 4 \end{bmatrix}$

8. Find the value of λ such that the following matrices are singular.

(i) $\begin{bmatrix} 2 & -3 & 5 \\ \frac{1}{2} & 4 & 6 \\ \lambda & 0 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 & 9 \\ 8 & \lambda & -8 \\ 2 & 2 & 4 \end{bmatrix}$

9. Solve each system. If the system has no solution, please state so.

(i)
$$x + y = 2$$
$$2x - z = 1$$
$$2y - 3z = -1$$

(ii)
$$2x - 6y + 3z = -12$$
$$3x - 2y + 5z = -4$$
$$2x + 5y - 2z = 10$$

(iii)
$$x - 2y + z = -1$$
$$3x + y - 2z = 4$$
$$y - z = 1$$

(iv)
$$2x + 2y + z = 1$$
$$x - y + 6z = 21$$
$$3x + 2y - z = -4$$

(v)
$$x + y - 2z = 3$$
$$3x - y + z = 5$$
$$3x + 3y - 6x = 9$$

10. Use Cramer's Rule to solve the system

(i)
$$x-2y+z=-1$$
$$3x+y-2z=4$$
$$y-z=1$$

(ii)
$$4x+8y+z=-6$$
$$2x-3y+2z=0$$
$$x+7y-3z=-8$$

(iii)
$$x+y-2z=2$$
$$3x-y+z=5$$
$$3x+3y-6z=6$$

(iv)
$$x-2y-2z=3$$
$$2x-4y+4z=1$$
$$3x-3y-3z=4$$

(v)
$$3x-2y+5z=6$$
$$4x-4y+3z=0$$
$$5x-4y+z=-5$$

(vi)
$$x+y+z=0$$
$$2x-y-4z=15$$
$$x-2y-z=7$$

Chapter 3

Groups

3.1 Binary Operations

We first define a mapping as follows:

Let A and B be two non-empty sets. Then a mapping f from A to B, written symbolically in the form $f: A \rightarrow B$, is a rule that assigns to each element $a \in A$ a unique element $b \in B$. We say b is the image of a under 'f' and write it symbolically in the form $f(a) = b$ or $b = f(a)$.

Let S be a non-empty set. A mapping $*: S \times S \rightarrow S$ is called a binary operation on S. Here we denote $*(a, b)$ by $a * b$.

Example 1

In the set Z of integers, the ordinary addition + of integers associates with each ordered pair (a, b) of integers a unique integer which we denote by $a + b$. Similarly the ordinary multiplication '$\cdot$' in Z associates with each ordered pair (a, b) of integers a unique integer which we denote by $a \cdot b$ or ab. Thus '+' and '$\cdot$' are binary operations on the set of integers.

Example 2

We similarly have binary operations of ordinary addition and multiplication defined on the set Q of rational numbers, the set R of real numbers and the set C of complex numbers.

Example 3

Consider the set N of natural numbers. If $a,b \in N$, then $a-b$ may or may not belong to N For example, $3,4 \in N$. Now $4-3 \in N$ but $3-4 \notin N$. Hence ' $-$ ' is not a binary operation on N.

Example 4

Similarly $\div$ is not a binary operation on N (or Z). Consider for example $2 \div 3$.

Example 5

The operation $+$ as defined for matrices is a binary operation on the set $M_{m \times n}$ of all $m \times n$ matrices over the field of real numbers (complex numbers).

Example 6

The multiplication '$\cdot$' as defined for matrices is a binary operation on the set M_n of all $n \times n$ matrices over the field of real numbers (complex numbers).

Binary operations are also defined with the help of a table as explained below:

Suppose $S = \{-1, 1\}$. We write the elements of the set in the first row and the first column of the table as shown below: The '$\cdot$' in the upper left hand corner indicates the binary operation on the set S. If a, b denote any two elements of the set, we write $a \cdot b$ or (ab) at the intersection of row facing a (on the left) and the column below b (on the top). Thus we have the following (multiplication) table.

•	1	-1
1	1	-1
-1	-1	1

Again suppose $S = \{1, -1, i, -i\}$. Then the following table defines the binary operation of multiplication on S. where $i = \sqrt{-1}$

•	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Now consider $S=\{0, 1\}$. Then the tables (I) and (II) given below define the binary operations of addition and multiplication on S.

+	0	1
0	0	1
1	1	0

(I)

•	0	1
0	0	0
1	0	1

(II)

3.2 Properties of Binary Operations

Let S be a non-empty set and $*$ be a binary operation on S.

$*$ is called *commutative* if $a * b = b * a$ for all $a, b \in S$.

$*$ is called *associative* if $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

An element $e \in S$ is called an *identity element* with respect to $*$ if $a * e = e * a = a$ for all $a \in S$.

An element $b \in S$ is called an *iverse* of $a \in S$ with respect to $*$ if $a * b = b * a = e$.

A set S cannot have more than one identity element with respect to a given binary operation, say $*$

Let us prove this.

Suppose S has two identity elements e and f with respect to $*$. Then

$e * f = f$ (considering 'e' as identity element)

Also $e * f = e$ (considering 'f' as identity element)

Hence $e = e * f = f$.

Thus identity element is unique and so we speak of *the identity* of a set with respect to a given binary operation.

If $*$ is associative, inverse of an element is unique. To prove this, suppose $b, c \in S$ are inverses of 'a' with respect to $*$. That is

$$a * b = b * a = e$$

$$a * c = c * a = e$$

Now $b * (a * b) = b * (a * c)$ $(\therefore a * b = e = a * c)$

$\Rightarrow (b * a) * b = (b * a) * c$ ($\therefore$ $*$ is associative)

$\Rightarrow e * b = e * c$ $\therefore$ $(b * a = e)$

$\Rightarrow b = c$ $\because$ (e is the identity element with respect to $*$)

This proves uniqueness of inverse element. We shall denote the unique inverse of 'a' by a^{-1}.

Examples

(1) The binary operations '+' and '·' on N are commutaitve and associative. There is no identity element with respect to +. However $1 \in N$ is the identity element with respect to '·'.

(2) The binary operations '+' and '·' on Z are commutative and associative. Here '0' is the identity element with respect to '+' and 1 is the identity element with respect to '·'. For each $a \in Z$, $-a$ is the inverse of a with respect to '+'. However, elements of Z except 1 and -1 do not possess inverses with respect to '·'.

(3) Both the operations '+' and '·' on Q are commutative and associative. The identity element with respect to '+' is 0 and the identity element with respect to '·' is 1. Each $a \in Q$ has an inverse with respect to '+'. Also each non-zero a in Q has an inverse with respect to '·'. The additive inverse of a is $-a$ and the multiplicative inverse of $a \in Q$, $a \neq 0$ is $\frac{1}{a}$ i.e $a^{-1} = \frac{1}{a}$

(4) We have results similar to (3) for the set R of real numbers and for the set C of complex numbers.

(5) For the set M_n of all $n \times n$ matrices over the field of real mumbers, addition '+' defined for matrices is

commutative and associative. The null matrix of order $n \times n$ is the identity element with respect to '+'. For each $n \times n$ matrix A, the matrix $-A$ is the inverse with respect to addition.

The multiplication '·' defined for matrices is associative but not commutative. The identity matrix I_n is the identity element with respect to '·'. Not every $n \times n$ matrix has an inverse with respect to multiplication.

Exercise 3.1

1. *Answer true or false.*
   (i) '+' is a binary operation on the set of all integers.
   (ii) '·' is a binary operation on the set of all positive real numbers.
   (iii) '+' is a binary operation on the set of all negative real numbers.
   (iv) ' − ' is a binary operation on the set of all negative real numbers.
   (v) '+' is a binary operation on the set of positive rational numbers.
2. *Answer true or false.*
   (i) A binary operation on a set S assigns exactly one element of S to each ordered pair of elements of S.
   (ii) A binary operation on a set S assgins at least

one element of S to each ordered pair of elements of S.

(iii) Both addition and subtraction are binary operations on the set N of natural numbers.

(iv) Subtraction is a commutative binary operation on the set R of real numbers.

(v) Addition is a binary operation on the set O of all odd integers.

3. Let $A = \{1, 2, 3, 4\}$. Define $*$ on A by the following table.

*	1	2	3	4
1	2	3	4	1
2	3	4	1	2
3	4	1	2	3
4	1	2	3	4

(i) Is $*$ commutative?

(ii) What is the identity element with respect to $*$

(iii) Write inverse of each element of A with respect to $*$.

4. Define a binary operation $*$ on the set R of real numbers by

$$a * b = a + b - a \cdot b$$

where '+', '·' and ' − ' are ordinary operations of addition, multiplication and subtraction defined for real numbers.

show that $*$ is commutative.

Find the identity element for the operation *.

5. Let R be the set of real numbers and let * be a binary operation on R defined by

   $$a * b = 2a + 3b$$

   for all $a, b \in R$. Show that * is not associative.

6. Let Z^+ denote the set of positive integers. Define an operation o on Z^+ by

   $$a \circ b = a + b + ab$$

   show that o is associative.

7. Let X be a non-empty set and $S = P(X)$ (power set of X). Define an operation * on P (X) by

   $A * B = A \cup B$ for all $A, B \in P(X)$.

   (i) Is * a binary operation on $P(X)$?

   (ii) Is * associative?

   (iii) Does there exist an identity element with respect to * ?.

8. Define * on $P(X)$ by $A * B = A \cap B$

   for all $A, B \in P(X)$.

   (i) Is * a binary operation on $P(X)$?

   (ii) Is * associative?

   (iii) Is there an identity element with respect to * ?

9. Is ordinary addition + defined on the set R of real numbers a binary operation on the set of negative real numbers? Is it associative?

10. Let $S = \{a, b, c, d\}$. Define * on S by the following

table.

*	a	b	c	d
a	d	a	c	b
b	a	c	b	d
c	b	d	a	c
d	c	b	d	a

Show that * is neither commutative nor associative. Show also that S has no identity element with respect to *. Why no element of S can have an inverse in this case?

3.3. Groups

An ordered pair $(G,.)$ consisting of a non-empty set G and a binary operation ‘·’ defined on G is called a *group*, if the following axioms are satisfied:

(i) The binary operation ‘·’ is associative.

(ii) G has an identity element e with respect to ‘·’.

(iii) For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a a^{-1} = a^{-1} a = e$ i.e. each $a \in G$ has inverse element a^{-1} in G.

G is called *abelian* or *commutative* group if ‘·’ is commutative.

If the binary operation is denoted by ‘+’, the identity element is denoted by O and the inverse of a w.r.t ‘+’ is denoted by $-a$.

Examples

The reader is already familiar with the properties of different number systems. It is now easy to check that

1. Neither $(N, +)$ nor $(N, .)$ is a group (why ?)
2. Neither $(W, +)$ nor $(W, .)$ is a group. (W denotes the set of all whole numbers).
3. $(Z, +)$ is a group but $(Z, .)$ is not a group.

To prove this, we note that '+' is a binary operation in Z.

'+' is associative in Z

0 (zero) is the identity element w.r.t '+'.

For each $a \in Z, -a \in Z$

$a + (-a) = (-a) + a = 0$

Hence all the axioms of a group are satisfied. So $(Z, +)$ is a group.

Now consider $(Z, .)$, we know that '·' is a binary operation in Z. '·' is associative in Z.

$1 \in Z$ and 1 is the identity with respect to '·'. If $a \in Z$, there exists no $b \in Z$ such that $ab = 1$.

Thus the axiom (iii) is not satisfied and so $(Z, .)$ is not a group.

4. Similarly it can be proved that $(Q, +)$ is a group and $(Q, .)$ is not a group becuase $0 \in Q$ and $0a = a0 = 0$ for all $a \in Q$. Thus O has no inverse with respect to multiplication (why?).

However, the set of all non-zero rational numbers i.e. $Q \setminus \{0\}$ is a group with respect to '·' (show that all the axioms are satisfied in this case).

5. As in example (4) it can be proved that $(R, +)$ and $(R\setminus\{0\}, .)$ are groups.
6. Similarly $(C, +)$ and $(C\setminus\{0\}, \cdot)$ are groups.
7. The set $M_{m\times n}$ of all $m \times n$ matrices over the field of real numbers form a group with respect to matrix addition '+'.

(Show that all the axioms for a group are satisfied).

8. The set M_n of all $n \times n$ matrices do not form a group under multiplication '·' defined for matrices. Why?

What about set of all $n \times n$ non-singular matrices? Do they form a group under matrix multiplication?

Remark 1

The groups of examples (1) to (7) are commutative (abelian). But the group in example (8) is not commutative because $AB \neq BA$, in general.

Remark 2

The groups of examples (1) to (8) are infinite groups. Following are examples of finite groups.

(9) Consider $(G, \cdot)$, where $G = \{1, \omega, \omega^2\}$ and '·' is defined by the table.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Then $(G, .)$ is a group with identity 1. It is a finite commutative group.

Here ω is read as omega and we shall see later that "ω is a cube root of unity".

10. Similarly the following tables define finite groups.

·	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	$+1$	-1

11.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

12.

∘	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

13. Consider singleton $\{e\}$ with binary operation '·' defined by $e \cdot e = e$. then $\{ e \}$ is a group w.r.t '·'.

Exercise 3.2

1. *Tick the correct answer.*

(a) To each element of a group, there corresponds (1) no (2) only one (3) more than one inverse element.

(b) The set Z of integers is

(1) finite group

(2) a group with respect to addition

(3) a group with respect to multiplication.

(c) The set of all non-zero rational numbers is

(1) a group with respect to '+'.

(2) a group with respect to '·'.

2. Is the following set a group and if so, state the binary operation of the group.

$$P = \{ a + i b, a, b \in Q \}$$

3. Is the set

$$A = \{ 1, -1 \}$$

a group under multiplication? Is it commutative? What is the inverse of -1 in A?

4. Prove that the set of complex numbers is an abelian group with respect to addition.

5. Show that $\{ 1, -1, i, -i \}$ is a group with respect to multiplication defined for complex numbers.

6. State clearly why the following are not groups?

(i) The set R of all real numbers with respect to division.

(ii) The set $Z \setminus \{o\}$ of all non-zero integers with respect to division.

7. Which of the following are true statements:

(i) A group can have more than one identity element.

(ii) Every set of numbers which is a group under addition is also a group under multiplication and vice-versa.

8. Let $G = \{a, b, e, 1\}$ with a binary operation $*$ defined by the following table:

$*$	a	b	e	1
a	a	b	1	e
b	b	a	e	1
e	1	e	a	b
1	e	1	b	a

Show that $(G, *)$ is not a group.

9. Let G be the set of negative integers and let $a * b$ be the number $a \cdot b$ for all a, b in G. Show that $(G, *)$ is not a group.

10. Let G be the set of all negative real numbers and let $a*b$ be the number $a + b$ for all a, b in G. Show that $(G,*)$ is a group but it is not abelian.

11. If $X = \{ a, b, c \}$, Find $P(X)$.

Is $(P(X), \cup)$ a group? Explain.

Is also $(P(X), \cap)$ a group?

12. Tick the correct answer.

Matrix multiplication (when defined) is

(i) not commutative but associative,

(ii) not commutative and not associative,

(iii) commutative and associative,

(iv) commutative but not associative.

13. A square matrix A has inverse if A is non-singular. Prove that the set of all 2×2 non-singular matrices forms a group with respect to matrix multiplication.

Chapter 4

Equations

4.1 Polynomials

An algebraic expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots\dots + a_1 x + a_0$$

where 'n' is any positive integer and $a_n, a_{n-1}, \dots\dots a_0$ are real numbers is called a polynomial of degree 'n'.

The real numbers $a_n, a_{n-1}, \dots\dots a_1, a_0$ are called the coefficients of the polynomial.

If $a_n \neq 0$, the polynomial is said to be of degree 'n'.

In particular if a polynomial has one, two or three terms, it is called *monomial*, *binomial* or *trinomial* respectively.

A polynomial of degree 'n' in x is denoted by $P_n(x)$.

In practice 'n' is omitted and we write $P(x)$ for any polynomial.

4.2 Remainder Theorem

If a polynomial $P_n(x)$ of degree $n \geq 1$ is divided by $x - r$, where 'r' is any constant, until a constant remainder is obtained then this remainder is $P_n(r)$.

Proof

Let $Q(x)$ be the quotient when $P_n(x)$ is divided by $x - r$ and let 'R' be the constant remainder.

As dividened = (divisor) · (quotient) + Remainder

$$\text{Therefore } P_n(x) = (x - r)\, Q(x) + R \qquad \ldots \quad (1)$$

Since (1) is true for all values of x, it is true in particular when $x = r$. Substituting r for x in (1) we get

$$P_n(r) = 0 \,.\, Q(r) + R$$

$$\text{or} \qquad P_n(r) = R$$

which proves the theorem.

Example 1

Without dividing, find the remainder when

$x^4 - x^3 - 9x + 5$ is divided by $x - 1$

Solution

Here $\quad P_4(x) = x^4 - x^3 - 9x + 5$ and $r = 1$

So $\quad P_4(1) = (1)^4 - (1)^3 - 9(1) + 5$

$= -4.$

Remainder $= -4$

Example 2

Show that if

$x^3 + 4x^2 - 7x - 3$ is divided by $x - 2$, the remainder is 7

Solution

Here $\quad P_3(x) = x^3 + 4x^2 - 7x - 3$

Remainder $= P_3(2)$

Therefore $P_3(2) = (2)^3 + 4(2)^2 - 7(2) - 3$

$= 8 + 16 - 14 - 3$

$= 7$

So Remainder = 7

4.3 Factor Theorem

If $P_n(r) = 0$, then $x - r$ is a factor of $P_n(x)$; or if 'r' is a root of $P_n(x) = 0$, then $x - r$ is a factor of $P_n(x)$.

If $P_n(x)$ is a polynomial, the values of x which make $P_n(x)=0$, are called the roots of the polynomial equation $P_n(x) = 0$.

Proof

As 'r' is a root of $P(x) = 0$ therefore $P(r) = 0$

By remainder theorem

$P(r) = R$, it follows that

$R = 0$

So $P(x) = (x - r)\,Q(x)$

This proves that $x - r$ is a factor of $P(x)$, while the other factor $Q(x)$ is the polynomial with degree one less than that of the degree of $P(x)$.

Converse of the Factor Theorem

If $x - r$ is a factor of $P_n(x)$, then $P_n(r) = 0$, or 'r' is a root of the polynomial equation $P_n(x) = 0$.

Proof

When $x - r$ is a factor of $P(x)$, then the remainder is zero when $P(x)$ is divided by $x - r$. Therefore, by remainder theorem

$P(r) = 0$

This proves that 'r' is a root of $P(x) = 0$

Example

Show by factor theorem that '2' is a root of the equation

$$x^3 - 6x^2 + 11x - 6 = 0$$

Solution

If '2' is a root of the equation, then $x - 2$ must be a factor of $P(x)$. In this case $P(2) = 0$

But $\quad P(2) = (2)^3 - 6(2)^2 + 11(2) - 6$

$$= 8 - 24 + 22 - 6$$

$$= 0$$

Which is true.

Exercise 4.1

Find the remainder by means of Remainder Theorem, when

1. $x^3 + 2x^2 + x + 1$ is divided by $x + 1$
2. $x^3 + x - 1$ is divided by $x + 1$
3. $x^4 - 2x^2 + 3x + 3$ is divided by $x - 3$
4. $x^4 + x^3 + 2x^2 - x + 7$ is divided by $x + 3$
5. Find the values of 'p' and 'q' if $x + 1$ and $x - 2$ are the factors of $x^3 + px^2 + qx + 6$.
6. If $x - 2$ and $x + 1$ are the factors of $x^3 + ax^2 + bx + 2$ then find the values of 'a' and 'b'.

7. For what value of 'k', $x - 3$ is a factor of the expression $2x^4 - 6x^3 - 7x^2 + kx - 15$.

By means of remainder theorem:

8. Show that '2' is a root of the equation
   $x^3 - 4x^2 + 5x - 2 = 0$

9. Prove that '3' is a root of the equation
   $x^3 - 6x^2 + 5x + 12 = 0$

10. For what value of 'k', the second polynomial is a factor of the first polynomial.
    (i) $x^3 - 5x^2 - 16x + k; x - 5$
    (ii) $x^3 - 9x^2 + 14x + k; x + 1$

4.4. Synthetic Division

It is a process of simplifying the division of a polynomial $P(x)$ by a binomial $x + a$. In the divisor the first term is always 'x'.

The method will be explained with the help of the following example.

Example

Divide $4x^4 - 5x^2 + 2x - 3$ by $x - 2$

Solution

First we write the dividened by arranging the terms in descending order, supplying each missing power with a zero as its coefficient. We write negative of the constant appearing in the divisor on the left hand corner as shown on the next page:

$$\begin{array}{r|rrrrrl} & 4 & 0 & -5 & 2 & -3 & \text{I line}\\ 2 & & 8 & 16 & 22 & 48 & \text{II line}\\ \hline & 4 & 8 & 11 & 24 & \multicolumn{1}{|r}{45} & \text{III line}\end{array}$$

The process employed is explained as follows:

The first line consists of the coefficients of the dividened after writing its terms in descending order then

1. '4' is brought down from the first line to the third line.
2. '8', the product of '2' and '4' is written in the next position in the second line below '0'.
3. '8', the sum of '0' and '8' is written in the third line.
4. '16', the product of '2' and '8' is written in the position in second line below – '5'.
5. '11', the sum of – '5' and '16' is written in the third line.
6. '22', the product of '2' and '11' is written in the second line below '2'.
7. '24', the sum of '2' and '22' is written in the third line.
8. '48', the product of '2' and '24' is written in the second line below – '3'.
9. '45', the sum of – '3' and '48' is written in the third line.

The last number in the third line is the remainder, and the other numbers in the third line are the coefficients of the powers of 'x' in the quotient arranged in descending powers of

'x'. The degree of the quotient will be one less than that of the degree of the dividened.

In the above case, the last number in the third line is '45'.

Thus, Quotient = $4x^3 + 8x^2 + 11x + 24$ and Remainder = 45

Example 1

Using synthetic division, find the remainder and the quotient when $2x^3 - 3x^2 + 4x + 1$ is divided by $x - 3$

Solution

First we write the coefficients of the dividened. The number on the left hand corner will be '3'. We proceed as follows:

	2	–3	4	1
3		6	9	39
	2	3	13	40

Therefore, Quotient = $2x^2 + 3x + 13$

Remainder = 40

Example 2

By use of synthetic division find $P(2)$ when

$$P(x) = x^3 + 3x^2 - 2x + 1$$

Solution

Here Divide $P(x)$ by $x - (+2) = x - 2$

	1	3	–2	1
2		2	10	16
	1	5	8	17

So, Remainder = 17

Hence $P(2) = 17$

Example 3

One root of $x^3 + x^2 - 10x + 8 = 0$ is '1'. Use synthetic division to find the other roots.

Solution

As '1' is a root of the given equation, therefore $x - 1$ will be a factor of the polynomial $x^3 + x^2 - 10x + 8$

For other factors using synthetic division

$$\begin{array}{c|cccc} & 1 & 1 & -10 & 8 \\ 1 & & 1 & 2 & -8 \\ \hline & 1 & 2 & -8 & \multicolumn{1}{|c}{0} \end{array}$$

Other factor is $x^2 + 2x - 8$

Other roots will be the roots of the equation

$x^2 + 2x - 8 = 0$

The equation can be written as

$(x + 4)(x - 2) = 0$

Hence $x = 2, -4$

Therefore the roots of the given equation are 1,2,– 4

Exercise 4.2

Use synthetic division to find the quotient and the remainder when the first polynomial is divided by the second polynomial in each case:

1. $3x^3 + 2x^2 - x - 1; x + 3$
2. $2x^3 - 7x^2 + 12x - 27; x - 3$
3. $x^3 - 3x^2 + x + 4; x - 1$

4. $x^3 + 2x^2 + 5x - 26; x - 2$

Find the values of the polynomials at the given value of 'x' by means of synthetic division:

5. $2x^3 + 3x^2 - 2x + 5; x = 2$
6. $3x^3 + 4x^2 - 5x + 6; x = -1$
7. $2x^4 - 5x^3 + 4x - 1; x = 1$
8. $4x^4 + 6x^3 + x^2 + x - 3; x = -2$

By means of synthetic division solve the equations, given that:

9. '3' is a root of $x^3 - 3x^2 + x - 3 = 0$
10. – '1' is a root of $x^3 - 5x^2 + 7x + 13 = 0$
11. – '1' and '2' are the roots of $x^4 - 5x^2 + 4 = 0$
12. – '1' is a double root of $x^4 - x^3 - 9x^2 - 11x - 4 = 0$

Use synthetic division to find the value of 'k' such that:

13. $x + 3$ is a factor of $3x^3 + kx^2 - 22x + 24$
14. – '2' is a root of $x^3 + 4x^2 + kx + 8 = 0$

4.5 Quadratic Equations

Quadratic equation or a second degree equation in 'x' is an integral rational equation in which the highest degree of the variable x is two. The standard form of a quadratic equation is

$$ax^2 + bx + c = 0$$

where a, b and c do not contain 'x' and are constants and $a \neq 0$; a and b are the coefficients of the second and first degree terms respectively, c is constant term. The standard form of the equation.

$$(2x-5)^2+(4x+3)^2=(3x+1)^2+15$$

on simplification is

$$11x^2-2x+18=0$$

A complete quadratic equation is one in which 'b' is non zero and a pure quadratic equation is one in which 'b' is zero. Usually we take a, b and c as rational numbers unless otherwise stated.

To solve a quadratic equation or to find the solution set of the equation means to determine the set of replacement for the variable which makes the equation true. The values of a variable which satisfy the equation are called the *Roots* of the equation.

4.6 Solution of a Quadratic Equation

A quadratic equation can be solved by any one of the following methods:

(i) Method of Factorisation

This method is based upon the property 'if in any number system $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$'.

In this method we first transform the given equation in the standard form of the quadratic equation. The left member of the equation is then factorised if possible. The factors thus obtained are equated to zero separately. The resultant equations are solved for the values of x. These values of 'x' give the solution of the equation.

Example

Solve $6x^2 + 5x = 6$

Solution

Write in standard form as

$$6x^2 + 5x - 6 = 0$$

Factorise the left member:

$$(3x - 2)(2x + 3) = 0$$

Equate each of the factors to zero

$$3x - 2 = 0 \text{ or } 2x + 3 = 0$$

Solve the equations for 'x'

$$x = \frac{2}{3} \text{ or } x = -\frac{3}{2}$$

Therefore

$$\text{Solution set} = \left\{-\frac{3}{2}, \frac{2}{3}\right\}$$

i.e the roots of the given equation are $-\frac{3}{2}, \frac{2}{3}$

Completing a Square

A binomial $x^2 + bx$ becomes a perfect square if we add the square of one half of the coefficient of x to it. We can complete the square if we add $\left(\frac{b}{2}\right)^2$ or $\frac{b^2}{4}$ to get

$$x^2 + bx + \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2$$

(ii) Finding Solution by the Method of Completing the Square:

To solve a quadratic equation in 'x' by completing a square:

1. Reduce the given equation in standard form.
2. Divide both sides by the coefficient of x^2, if it is other than '1'.
3. Transfer the term independent of 'x' to the right hand side of the equation.
4. Add the square of one half of the coefficient of 'x' to both members of the equation.
5. Rewrite the left member as the square of a binomial and simplify the right member.
6. Extract square root of each member using plus and minus signs on the right and solve for 'x'.

Example

Solve by completing a square:

$$3x^2 - 8x + 4 = 0$$

Solution

The equation is given in standard form as

$$3x^2 - 8x + 4 = 0$$

Transpose '4' to the right

$$3x^2 - 8x = -4$$

Divide by 3: $x^2 - \frac{8}{3}x = -\frac{4}{3}$

Add $\left(\frac{4}{3}\right)^2$: $x^2 - \frac{8}{3}x + \frac{16}{9} = -\frac{4}{3} + \frac{16}{9}$

$$\left(x - \frac{4}{3}\right)^2 = \frac{-12 + 16}{9}$$

or $\left(x - \frac{4}{3}\right)^2 = \frac{4}{9}$

Extract square roots using plus and minus signs in the right member

So $x - \frac{4}{3} = \pm\frac{2}{3}$

If $x - \frac{4}{3} = \frac{2}{3}$ then $x = 2$, and

If $x - \frac{4}{3} = -\frac{2}{3}$ then $x = \frac{2}{3}$

Hence roots are $2, \frac{2}{3}$

Solution set = $\left\{2, \frac{2}{3}\right\}$

(iii) The Quadratic Formula

The quadratic equation is

$ax^2 + bx + c = 0.\ a \neq 0$

Transpose 'c': $ax^2 + bx = -c$

Divide by a : $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Add $\left(\frac{b}{2a}\right)^2$ or $\frac{b^2}{4a^2}$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

Simplify : $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

Extracting square roots

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

or $\quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \cdots \quad (A)$$

We call (A) the *quadratic formula*. In this formula it is permissible for a, b and c to have any values with $a \neq 0$.

To solve a quadratic equation by the quadratic formula, the equation is first reduced to the standard form, then the formula is applied by putting the values of a, b and c given in the equation.

Example

Solve $15x^2 = 4x + 4$, using quadratic formula.

Solution

$15x^2 - 4x - 4 = 0$

Here $a = 15, b = -4, c = -4$

From the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For the given equation.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\,(15)\,(-4)}}{2(15)}$$

$$= \frac{4 \pm \sqrt{16 + 240}}{30}$$

$$= \frac{4 \pm \sqrt{256}}{30}$$

$$= \frac{4 \pm 16}{30}$$

$$= \frac{20}{30}, -\frac{12}{30}$$

Thus $x = \frac{20}{30} = \frac{2}{3}$ and $x = -\frac{12}{30} = -\frac{2}{5}$

Solution set $= \left\{\frac{2}{3}, -\frac{2}{5}\right\}$

Exercise 4.3

Solve the following equations by use of factors:

1. $x^2 + 5x - 6 = 0$
2. $x^2 - 3x + 2 = 0$
3. $8x^2 - 2x - 3 = 0$
4. $4x^2 + 8x - 5 = 0$
5. $2x^2 - 3x + 1 = 0$
6. $2x^2 + 3x - 9 = 0$
7. $6x^2 + 11\,x - 10 = 0$

8. $6 - x - 2x^2 = 0$

9. $8x^2 - 15 - 2x = 0$

10. $15x^2 + x - 6 = 0$

Solve for 'x' by completing a square.

11. $x^2 - 3x + 18 = 0$

12. $6x^2 - 13x + 6 = 0$

13. $5x^2 - 2x - 3 = 0$

14. $4x^2 - 25 = 48x$

15. $3x^2 - 5x - 2 = 0$

16. $6 + 17x + 12x^2 = 0$

17. $7x^2 - 17x + 6 = 0$

18. $5x^2 + 23x + 12 = 0$

19. $6x^2 - 20 + 7x = 0$

20. $2 + 5x - 3x^2 = 0$

21. Solve questions from 1 to 20 by using quadratic formula. Also check the results by putting the values of the roots in the given equations.

4.7 Equations Reducible to the Quadratic Form

There are other types of equations which are not quadratics but can be reduced to quadratic forms by suitable transformations, then they can be easily solved. As a result of these transformations sometimes new roots are introduced. These roots do not satisfy the original equations. These roots which do not satisfy the equation are called *Extraneous roots* of the equation. The different types of equations are introduced as under:-

Type I: Equations Involving Fractions

Example 1

Solve $\dfrac{3}{x+1}+\dfrac{4}{x+2}=2$

Solution

$$\frac{3}{x+1}+\frac{4}{x+2}=2$$

Multiply both sides of the equation by $(x + 1)(x + 2)$ to get

$$3(x + 2) + 4(x + 1) = 2(x + 1)(x + 2)$$

$$3x + 6 + 4x + 4 = 2x^2 + 6x + 4$$

Writing in standard form

$$2x^2 - x - 6 = 0$$

or $(2x + 3)(x - 2) = 0$

Hence $2x + 3 = 0$ or $x - 2 = 0$

Thus $x = -\dfrac{3}{2}$ or $x = 2$

Solution set $= \left\{2,\ -\dfrac{3}{2}\right\}$

Type II. Equations of the Type $ay + \dfrac{b}{y} = c$

Example 2

Solve $\sqrt{x+3}+\dfrac{1}{\sqrt{x+3}}=\dfrac{5}{2}$

Solution

$$\sqrt{x+3}+\frac{1}{\sqrt{x+3}}=\frac{5}{2}$$

Let $\sqrt{x+3}=y$. The equation can be written in the form

$$y+\frac{1}{y}=\frac{5}{2}$$

or $$2y^2-5y+2=0$$

Factorising

$(2y-1)(y-2)=0$

Thus $2y-1=0$ or $y-2=0$

Hence $y=\frac{1}{2}$ or $y=2$

If $y=\frac{1}{2}$, we have

$\sqrt{x+3}=\frac{1}{2}$

Squaring $x+3=\frac{1}{4}$

$x=\frac{1}{4}-3=-\frac{11}{4}$

If $y=2$, then

$\sqrt{x+3}=2$

Squaring

$x+3=4$

or $x=1$

Hence solution set $=\left\{1,\ -\frac{11}{4}\right\}$

Type III. Equations of the Type $ax^{2n} + bx^n + c = 0$

Example 3

Solve $x^4 - 3x^2 - 4 = 0$

Solution

$x^4 - 3x^2 - 4 = 0$

Let $x^2 = y$

The equation can be written in the form

$y^2 - 3y - 4 = 0$

Factorising: $(y - 4)(y + 1) = 0$

$y - 4 = 0$ or $y + 1 = 0$

Therefore $y = 4$ or $y = -1$

If $y = 4$, then $x^2 = 4$

So $x = \pm 2$

When $y = -1$, then $x^2 = -1$

Thus $x = \pm i$

Solution set $= \{\pm 2, \pm i\}$

Type IV Irrational Equations

Example 4

Solve $\sqrt{x+2} + \sqrt{x-3} = \sqrt{4x-3}$

Solution

$\sqrt{x+2} + \sqrt{x-3} = \sqrt{4x-3}$

Squaring both sides to get.

$x + 2 + x - 3 + 2\sqrt{(x+2)(x-3)} = 4x - 3$

Grouping terms as

$$2\sqrt{(x+2)(x-3)} = 2x-2$$

$$\text{or} \quad \sqrt{(x+2)(x-3)} = x-1$$

Squaring again

$$(x+2)(x-3) = (x-1)^2$$

$$\text{or} \quad x^2 - x - 6 = x^2 - 2x + 1$$

$$x = 7$$

Hence solution set = {7}

Type V: Equations of the Type

$$l(ax^2 + bx) + m\sqrt{ax^2 + bx + c} = n$$

Example 5

Solve $x^2 + 3x + 3\sqrt{x^2 + 3x + 3} = 1$

Solution

$$x^2 + 3x + 3\sqrt{x^2 + 3x + 3} = 1$$

Let $x^2 + 3x + 3 = y^2$

The equation takes the form

$$y^2 - 3 + 3y = 1 \quad \text{as } x^2 + 3x = y^2 - 3$$

$$y^2 + 3y - 4 = 0$$

$$\text{or} \quad (y+4)(y-1) = 0$$

Then $y + 4 = 0$ or $y - 1 = 0$

If $y + 4 = 0$, then $y = -4$

Then $x^2 - 3y + 3 = 16$

$$x^2 - 3y - 13 = 0$$

Using quadratic formula

$$x = \frac{3 \pm \sqrt{9+52}}{2} = \frac{3 \pm \sqrt{61}}{2}$$

When $y - 1 = 0$, then $y = 1$

Hence $\quad x^2 + 3x + 3 = 1$

or $\quad x^2 + 3x + 2 = 0$

$$(x+1)(x+2) = 0$$

$x = -1, -2$

Solution set = $\left\{-1, -2, \frac{3 \pm \sqrt{61}}{2}\right\}$

Type VI: Exponential form

Example 6

Solve $4^{1+x} + 4^{1-x} = 10$

Solution

$$4^{1+x} + 4^{1-x} = 10$$

The equation can be written as

$$4 \cdot 4^x + 4 \cdot 4^{-x} = 10$$

Let $4^x = y$

The equation reduces to

$$4 \cdot y + \frac{4}{y} = 10$$

or $\quad 4y^2 - 10y + 4 = 0$

or $\quad 2y^2 - 5y + 2 = 0$

Factorising

$$(2y-1)(y-2) = 0$$

$2y - 1 = 0$ or $y - 2 = 0$

If $2y - 1 = 0$ then $y = \frac{1}{2}$

Therefore $4^x = \frac{1}{2}$

$(2)^{2x} = (2)^{-1}$

$2x = -1$ or $x = -\frac{1}{2}$

If $y - 2 = 0$, then $y = 2$

So $(2)^{2x} = (2)^1$

$2x = 1$ or $x = \frac{1}{2}$

Solution set $= \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

Type VII $\sqrt{x^2 + bx + c} \pm \sqrt{x^2 + bx + c'} = d$

Example 7

Solve $\sqrt{3x^2 - 4x + 34} + \sqrt{3x^2 - 4x - 11} = 9$

Solution

Let $a = \sqrt{3x^2 - 4x + 34}$ and $b = \sqrt{3x^2 - 4x - 11}$

Then we are given that $a + b = 9$ ... (1)

But $a^2 - b^2 = \left(3x^2 - 4x + 34\right) - \left(3x^2 - 4x - 11\right)$

$= 45$ ... (2)

From (1) and (2) by division

$$\frac{a^2 - b^2}{a + b} = \frac{45}{9}$$

or $$\frac{(a-b)(a+b)}{a+b}=5$$

or $$a-b=5 \qquad \cdots \qquad (3)$$

From (1) and (3) by addition

$$2a=14$$

or $$a=7$$

When $a = 7$, then $b = 2$

Therefore $\sqrt{3x^2-4x+34}=7$

Squaring

$$3x^2-4x+34=49$$

$$3x^2-4x-15=0$$

or $$(3x+5)(x-3)=0$$

$$x=-\frac{5}{3}, x=3$$

Also $\sqrt{3x^2-4x-11}=2$

Squaring

$$3x^2-4x-11=4$$

$$3x^2-4x-15=0$$

Again $x=-\frac{5}{3}, x=3$

Solution set $=\left\{3,-\frac{5}{3}\right\}$

Type VIII Common Factor and its Removal where possible

Example 8

Solve the equation

$$\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$$

Solution

$$\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$$

Factorising the expressions under the radical signs, we have

$$\sqrt{(x+7)(x-3)}+\sqrt{(x-3)(x+2)}=\sqrt{(6x+13)(x-3)}$$

$$\sqrt{x-3}\,[\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}]=0$$

Thus either $\sqrt{x-3}=0$ *i.e* $x=3$

or $\sqrt{x+7}+\sqrt{x+2}-\sqrt{6x+13}=0$

If $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

Then by squaring

$$x+7+x+2+2\sqrt{(x+7)(x+2)}=6x+13$$

or $4x+4=2\sqrt{(x+7)(x+2)}$

or $2x+2=\sqrt{(x+7)(x+2)}$

Squaring again, we have

$$(2x+2)^2=(x+7)(x+2)$$

$$4x^2+8x+4=x^2+9x+14$$

or $3x^2-x-10=0$

Therefore, $x=\dfrac{1\pm\sqrt{1+120}}{6}$

$$= \frac{1 \pm 11}{6}$$

$$= 2, -\frac{5}{3}$$

On checking it is observed that $x = 3, 2$ satisfy the equation but $x = -\frac{5}{3}$ does not. Thus the solution set is $\{3, 2\}$.

Type IX Equation of the Type

$$(x + a)(x + b)(x + c)(x + d) = k$$

where $a + b = c + d$

Example 9

Solve $(x + 1)(x + 2)(x + 3)(x + 4) = 24$

Solution

$$(x + 1)(x + 2)(x + 3)(x + 4) = 24$$

Re-arranging the factors on the left we have

$$[(x + 1)(x + 4)][(x + 2)(x + 3)] = 24$$

or $\quad (x^2 + 5x + 4)(x^2 + 5x + 6) = 24$

Let $\quad x^2 + 5x = y$

then $\quad (y + 4)(y + 6) = 24$

or $\quad y^2 + 10y + 24 = 24$

or $\quad y^2 + 10y = 0$

or $\quad y(y + 10) = 0$

Therefore $y = 0$ and $y = -10$

If $y = 0$, then $x^2 + 5x = 0$

Hence $x = 0, -5$

When $y = -10$, $x^2 + 5x = -10$

or $\quad x^2 + 5x + 10 = 0$

$$x = \frac{-5 \pm \sqrt{25-40}}{2}$$

$$x = \frac{-5 \pm \sqrt{-15}}{2}$$

Hence the solution set of the given equation is $\left\{0, -5, \frac{-5 \pm \sqrt{-15}}{2}\right\}$

Type X Reciprocal Equations

Equations of the form $ax^4 + bx^3 + cx^2 + bx + a = 0$, in which the coefficients from the begining and end are equal, are called reciprocal equations.

Example 10

Solve $8x^4 - 6x^3 + 7x^2 - 6x + 8 = 0$

Solution

$$8x^4 - 6x^3 + 7x^2 - 6x + 8 = 0$$

We write the given equation in the form

$$8(x^4 + 1) - 6(x^3 + x) + 7x^2 = 0$$

Dividing by x^2

$$8\left(x^2 + \frac{1}{x^2}\right) - 6\left(x + \frac{1}{x}\right) + 7 = 0$$

Let $x + \frac{1}{x} = y$

So

$$x^2+\frac{1}{x^2}+2=y^2$$

or $$x^2+\frac{1}{x^2}=y^2-2$$

Thus the given equation reduces to

$$8(y^2-2)-6y+7=0$$

or $$8y^2-16-6y+7=0$$

or $$8y^2-6y-9=0$$

or $$(4y+3)(2y-3)=0$$

If $4y+3=0$, then $y=-\frac{3}{4}$

Hence $$x+\frac{1}{x}=-\frac{3}{4}$$

or $$4x^2+3x+4=0$$

$$x=\frac{-3\pm\sqrt{9-64}}{8}$$

$$=\frac{-3\pm\sqrt{-55}}{8}$$

If $2y-3=0$, then $y=\frac{3}{2}$

In this case

$$x+\frac{1}{x}=\frac{3}{2}$$

or $$2x^2-3x+2=0$$

$$x=\frac{3\pm\sqrt{9-16}}{4}$$

$$=\frac{3\pm\sqrt{-7}}{4}$$

Solution set $=\left\{\frac{-3\pm\sqrt{-55}}{8}, \frac{3\pm\sqrt{-7}}{4}\right\}$

Type XI Equations of the Type $\frac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=k$

Example 11

Solve $\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=\frac{1}{2}$.

Solution

$$\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}=\frac{1}{2}$$

Applying componendo and dividendo, we have

$$\frac{(\sqrt{x+1}-\sqrt{x-1})+(\sqrt{x+1}+\sqrt{x-1})}{(\sqrt{x+1}-\sqrt{x-1})-(\sqrt{x+1}+\sqrt{x-1})}=\frac{1+2}{1-2}$$

or $$\frac{2\sqrt{x+1}}{-2\sqrt{x-1}}=\frac{3}{-1}$$

or $$\frac{\sqrt{x+1}}{\sqrt{x-1}}=3$$

Squaring

$$\frac{x+1}{x-1}=9$$

or $$x+1=9x-9$$

or $$8x=10$$

or $$x=\frac{10}{8}=\frac{5}{4}.$$

Solution set $=\left\{\frac{5}{4}\right\}$

Exercise 4.4

Solve the following equations.

1. $x - 6\sqrt{x} + 8 = 0$

2. $\left(x + \frac{1}{x}\right)^{-2} + 2\left(x + \frac{1}{x}\right)^{-1} - 8 = 0$

3. $2x^4 - 17x^2 - 9 = 0$

4. $x^{2/3} - 2x^{1/3} - 8 = 0$

5. $\left(\frac{x+2}{x-1}\right)^2 + \left(\frac{x+2}{x-1}\right) - 6 = 0$

6. $\frac{x-2}{x+4} + \frac{x+4}{x-2} = \frac{17}{4}$

7. $\frac{2x+3}{3x-4} + \frac{3x-4}{2x+3} = \frac{53}{14}$

8. $\sqrt{\frac{x}{x-3}} + \sqrt{\frac{x-3}{x}} = \frac{5}{2}$

9. $\sqrt{\frac{x}{x+2}} + 5 = 14\sqrt{\frac{x+2}{x}}$

10. $\sqrt{\frac{x+3}{x-4}} + 12\sqrt{\frac{x-4}{x+3}} = 7$

11. $\sqrt{3x+1} + \frac{1}{\sqrt{3x+1}} = \frac{17}{4}$

12. $2x^2 - 3x - 5\sqrt{2x^2 - 3x + 2} + 6 = 0$

13. $2x^2 - 4x - \sqrt{x^2 - 2x + 13} = 2$

14. $x^2 - x + 3\sqrt{x^2 - 3x + 2} = 2(x-1)$

15. $x + \sqrt{x^2 - 5x + 9} = \frac{x^2}{5} + 3$

16. $\sqrt{2x+1} + \sqrt{x} = \sqrt{6x+1}$

17. $\sqrt{x+3}+\sqrt{x}=\sqrt{x+8}$

18. $\sqrt{3x+4}+\sqrt{5-x}=5$

19. $\sqrt{x^2+4x+5}+\sqrt{x^2+4x+1}=2$

20. $\sqrt{x^2+3x+7}-\sqrt{x^2+3x+1}=3$

21. $(x^2-x)+3\sqrt{x^2-3x+2}=2(x-1)$

22. $(x+1)(x+2)(x+3)(x+4)=360$

23. $x^4+4x^3+5x^2+4x+1=0$

24. $\left(x-\frac{1}{x}\right)^2-8\left(x+\frac{1}{x}\right)-16=0$

25. $\left(x+\frac{1}{x}\right)^2-6\left(x-\frac{1}{x}\right)-11=0$

26. $\sqrt{2x^2+5x+13}+\sqrt{x^2+5x+4}=x+3$

27. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=5$

28. $3^{1+x}+4\cdot 3^{-x}-7=0$

29. $2^{1+x}-3\cdot 2^{1-x}-1=0$

30. $x^4-6x^2+10-\frac{6}{x^2}+\frac{1}{x^4}=0$

4.8 *Simultaneous Equations*

Equations having common solution are called simultaneous equations.

Consider a system of equations

$$\left.\begin{aligned} f(x,y)&=0\\ g(x,y)&=0\end{aligned}\right\}\quad\cdots\qquad(1)$$

where $f(x, y)$ and $g(x, y)$ are two polynmials in two variables x and y. The solution set of the system (1) is the set of all ordered pairs (x, y), which satisfy both the given equations of the system.

If S_1 is the solution set of the equation

$f(x,y) = 0$ and S_2 is the solution set of $g(x, y) = 0$, then the solution set of the given system is $S_1 \cap S_2$.

The process of finding these values of x and y, which satisfy both the equations is called solving these equations simultaneously for x and y.

The general equation $ax + by + c = 0$, where a, b, c are constants and a, b are not simultaneously zero, is said to be a linear or first degree equation in two unknowns x and y. A solution of this equation is an ordered pair (x_1, y_1), such that

$ax_1 + by_1 + c = 0$

4.9 *Solution of Simultaneous Equations*

In this section we shall try to review the usual method of determining a solution set of a pair of equations.

Type I when one equation is linear and the other is quadratic.

We proceed as:

(i) Find the value of one variable in terms of the other from the linear equation, say

$y = m\,x + c$

(ii) Substitute $mx + c$ for y in the second equation, thus getting the equation containing the variable x only.

(iii) Solve the equation obtained in (ii) for x.

(iv) Substitute the values of x in the equation

$y = m\,x + c$ to get the corresponding values of y.

(v) The values of x and y put in the form of ordered pairs give solution set of the equations.

Example 1

Solve the system

$$2x + y = 10 \quad \ldots \quad (1)$$

$$4x^2 + y^2 = 68 \quad \ldots \quad (2)$$

We use (1) to express y in terms of x and obtain

$$y = 10 - 2x \quad \ldots \quad (3)$$

Substituting this value of 'y' in (2) we have

$$4x^2 + (10 - 2x)^2 = 68$$

Which reduces to

$$4x^2 + 100 + 4x^2 - 40\,x - 68 = 0$$

or $$8x^2 - 40x + 32 = 0$$

Dividing by 8 to get

$$x^2 - 5x + 4 = 0$$

or $$(x - 4)\,(x - 1) = 0$$

If $x - 4 = 0$, then $x = 4$,

If $x - 1 = 0$, then $x = 1$

Substituting these results in (3) we get for $x = 4$, $y = 2$ and for $x = 1, y = 8$. Both solutions should be checked by substituting in both the equations. The solution set is $\{(4, 2), (1, 8)\}$

Type II Systems of the Form $Ax^2 + By^2 = C$

Two simultaneous equations of this form should first be solved as linear equations in x^2 and y^2. The values of x and y can then be found by extracting the square roots of the values of x^2 and y^2

Example 2

Solve

$$x^2 + y^2 = 13 \quad \ldots \quad (1)$$

$$3x^2 + 2y^2 = 30 \quad \ldots \quad (2)$$

Solution

Multiply (1) by 2 to get

$$2x^2 + 2y^2 = 26 \quad \ldots \quad (3)$$

Substract (3) from (2)

$$x^2 = 4$$

or $\quad x = \pm 2$

Putting $x = 2$ in (1) we get $y = \pm 3$

Putting $x = -2$ in (1) we get $y = \pm 3$

Values of x paired with values of y, the solution set is $\{(2, 3), (2, -3), (-2, 3), (-2, -3)\}$

Type IIISystems of the Type $Ax^2 + Bxy + Cy^2 = D$

Example 3

Solve

$$2x^2 - xy + y^2 = 4 \quad \ldots \quad (1)$$

$$4x^2 - 5xy + 3y^2 = 6 \quad \ldots \quad (2)$$

Solution

We first eliminate the constant terms from (1) and (2)

To eliminate the constants we multiply (1) by 3 and (2) by 2

$$6x^2 - 3xy + 3y^2 = 12 \quad \ldots \quad (3)$$

$$8x^2 - 10xy + 6y^2 = 12 \quad \ldots \quad (4)$$

Subtracting (4) from (3)

$$-2x^2 + 7xy - 3y^2 = 0$$

or $$2x^2 - 7xy + 3y^2 = 0$$

This may be factorised as

$(2x - y)(x - 3y) = 0$

Now we can solve either of the given equations together with both the equations obtained by putting the factors found above equal to zero. Thus we may solve the systems

$$\left.\begin{aligned} 2x - y &= 0 \\ 2x^2 - xy + y^2 &= 4 \end{aligned}\right\} \quad \ldots \quad (5)$$

and

$$\left.\begin{aligned} x - 3y &= 0 \\ 2x^2 - xy + y^2 &= 4 \end{aligned}\right\} \quad \ldots \quad (6)$$

Solving (5) we have $y = 2x$

Putting this value of y in the second equation in (5)

$$2x^2 - 2x^2 + 4x^2 = 4$$

or $$4x^2 = 4$$

$$x^2 = 1$$

$$x = \pm 1$$

and $y = \pm 2$

Solving (6) we get $x = 3y$

$$18y^2 - 3y^2 + y^2 = 4$$

$$16y^2 = 4$$

$$y^2 = \frac{1}{4}$$

$$y = \pm\frac{1}{2}$$

$$\text{and } x = \pm\frac{3}{2}$$

Solution set is $\left\{(1, 2), (-1, -2), \left(\frac{3}{2}, \frac{1}{2}\right), \left(-\frac{3}{2}, -\frac{1}{2}\right)\right\}$

Type IV When both the equations are quadratic and one of them is homogeneous equation

An equation is said to be homogeneous if every term in the equation is of the same degree.

Example 4

Solve

$$x^2 + y^2 = 14 \quad \ldots \quad (1)$$

$$x^2 - 3xy + 2y^2 = 0 \quad \ldots \quad (2)$$

Solution

Equation (2) can be written as

$$x^2 - 2xy - xy + 2y^2 = 0$$

$$x(x-2y)-y(x-2y)=0$$

$$(x-y)(x-2y)=0$$

Now we can solve either of the given equations together with both the equations obtained by putting these factors each equal to zero. Thus we may solve the systems

$$\left.\begin{aligned} x-y&=0\\ x^2+y^2&=14 \end{aligned}\right\} \quad \cdots \quad (3)$$

and

$$\left.\begin{aligned} x-2y&=0\\ x^2+y^2&=14 \end{aligned}\right\} \quad \cdots \quad (4)$$

Solving (3) $\qquad x=y$

Putting this value of x in the second equation in (3)

$$2y^2=14$$

or $\quad y^2=7$

$$y=\pm\sqrt{7}$$

and $x=\pm\sqrt{7}$

Solving (4) $\qquad x=2y$

Putting $x=2y$ in the second equation in (4) we have

$$4y^2+y^2=14$$

$$5y^2=14$$

$$y^2=\frac{14}{5}$$

or $\qquad y=\pm\sqrt{\frac{14}{5}}$

Hence $\quad x=\pm 2\sqrt{\frac{14}{5}}$

Solution set is $\left\{(\pm\sqrt{7}, \pm\sqrt{7}), \left(\pm 2\sqrt{\frac{14}{5}}, \pm\sqrt{\frac{14}{5}}\right)\right\}$

Type V Special Devices for Reducing to Simpler Form

Example 5

Solve

$$x^3 + y^3 = 27 \quad \ldots \quad (1)$$

$$x + y = 3 \quad \ldots \quad (2)$$

Solution

From (1)

$$(x + y)(x^2 - xy + y^2) = 27$$

$$3(x^2 - xy + y^2) = 27$$

$$\text{or} \quad x^2 - xy + y^2 = 9 \quad \ldots \quad (3)$$

The students should find the solution by using method as in type I.

Exercise 4.5

Solve the following simultaneous equations:

1. $2x - y + 15 = 0$
   $x^2 + y^2 = 45$
2. $x + 3y - 7 = 0$
   $3x^2 + 2y^2 = 11$
3. $3x - 5y - 3 = 0$
   $xy = 12$
4. $x - y - 7 = 0$
   $x^2 + 3xy + y^2 = 4$

5. $2x + 3y = 8$

   $2x^2 + 3y^2 = 14$

6. $3x + y = 5$

   $x^2 + xy = 3$

7. $4x + 3y = 25$

   $\frac{4}{x} + \frac{3}{y} = 2$

8. $x + y = 7$

   $\frac{5}{x} + \frac{2}{y} = 2$

9. $x - y = a - b$

   $\frac{a}{x} + \frac{b}{y} = 2$

10. $x + 2y - 4 = 0$

    $y^2 - x - 4 = 0$

11. $2x^2 - 3y^2 = 5$

    $4x^2 + 9y^2 = 25$

12. $x^2 - y^2 = 4$

    $2x^2 + y^2 = 11$

13. $3x^2 + 4y^2 = 43$

    $2x^2 - y^2 = 14$

14. $7x^2 - 3y^2 = 7$

    $11x^2 - 5y^2 = 10$

15. $5x^2 + 3y^2 = 20$

    $x^2 + y^2 = 4$

16. $x^2 + y^2 + 6x - 8y = 1$

$x^2 + y^2 + 4x - 7y = 4$

17. $x^2 + y^2 - 8x = 0$

$x^2 + y^2 - 4x - 3y = 4$

18. $x^2 + y^2 + 6x - 4y = 4$

$x^2 + y^2 + 11x - 5y = 6$

19. $x^2 + y^2 + 4x - 2y + 1 = 0$

$(x - 4)^2 + (y + 7)^2 = 64$

20. $(x - 1)^2 + (y + 2)^2 = 8$

$x^2 + y^2 - 3x + y = 12$

21. $2x^2 - xy + y^2 = 4$

$2x^2 - 7xy + 3y^2 = 0$

22. $2x^2 + 5xy - 3y^2 = 0$

$2x^2 + 3xy = 2$

23. $x^2 - 5xy + 6y^2 = 10$

$x^2 - xy = 4$

24. $4x^2 - 5xy + y^2 = 10$

$3x^2 - 3xy - y^2 = 5$

25. $2x^2 - xy = 12$

$xy + y^2 = 6$

26. $x^2 - y^2 = 27$

$x + y = 3$

27. $x^2 + xy + y^2 = 13$

$x^3 - y^3 = 26.$

28. $x^3 + y^3 = 9$

$x^2 - xy + y^2 = 3$

29. $x^2 + xy + y^2 = 19$

$xy = 6$

30. $\frac{1}{x^2} - \frac{3}{y^2} = 1$

$\frac{2}{x^2} - \frac{1}{y^2} = 7$

4.10 Nature or Character of the Roots of a given Quadratic Equation

We shall determine some useful facts about the roots of the equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

without actually solving the equation. If α, β are the roots of this equation then by quadratic formula, we know that

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Let a, b, c be the real numbers with $a \neq 0$. We shall consider the expression $b^2 - 4ac$ which appears under the radical sign in the formula.

The nature or character of the roots depends upon $b^2 - 4ac$, which is called the *Discriminant* of the quadratic equation. If we know the value of $b^2 - 4ac$, we can tell the general character or nature of the roots without solving the equation.

1. If $b^2 - 4ac$ is positive, then $\sqrt{b^2 - 4ac}$ is a real number

and the two roots of the equation are real numbers. Moreover the roots will be unequal.

2. If $b^2 - 4ac = 0$, then the roots are real and equal and are each equal to $-\frac{b}{2a}$

3. If $b^2 - 4ac$ is negative then the roots are complex and unequal.

In addition to above the roots of the equation will be rational numbers if and only if, $\sqrt{b^2 - 4ac}$ is rational, that is, if $b^2 - 4ac$ is a perfect square and

the roots will be irrational if $b^2 - 4ac$ is positive but not a perfect square.

We can summarize as follows:

If a, b, c are real numbers with $a \neq 0$, then the roots of $ax^2 + bx + c = 0$ are

(i) **real and unequal if $b^2 - 4ac$ is positive**

(ii) **real and equal if $b^2 - 4ac = 0$**

(iii) **imaginary (complex) and unequal if $b^2 - 4ac$ is negative.**

(iv) **rational if $b^2 - 4ac$ is a perfect square.**

(v) **irrational if $b^2 - 4ac$ is not a perfect square.**

Example 1

Find the nature of the roots of $2x^2 - 5x + 2 = 0$

Solution:

$$2x^2 - 5x + 2 = 0$$

Here $a = 2, b = -5, c = 2$

The coefficients are rational numbers.

$$b^2 - 4ac = (-5)^2 - 4\,(2)\,(2) = 9 = 3^2$$

is a perfect square. Hence the roots are real, rational and unequal.

Example 2

Find the nature of the roots of the equation

$$3x^2 + 7x - 2 = 0$$

Solution

$$3x^2 + 7x - 2 = 0$$

$$\text{Discriminant} = (7)^2 - 4\,(3)\,(-2)$$

$$= 49 + 24 = 63 > 0$$

which is positive but not a perfect square. Hence the roots are real, irrational and unequal.

Example 3

Find the nature of the roots of the equation

$$3x^2 + 4x + 5 = 0$$

Solution

$$3x^2 + 4x + 5 = 0$$

$$\text{Discriminant} = (4)^2 - 4\,(3)\,(5)$$

$$= 16 - 60 = -44 < 0$$

which is negative.

Hence the roots are imaginary and unequal.

Example 4

For what value of 'm' the roots of the equation $(3m+1)x^2+(m+3)x+(m-4)=0$ are equal.

Solution

$$(3m+1)x^2+(m+3)x+(m-4)=0$$

Here $a=3m+1,\ b=m+3,\ c=m-4$

$$b^2-4ac=(m+3)^2-4(3m+1)(m-4)$$

The roots are equal if $b^2-4ac=0$

So $(m+3)^2-4(3m+1)(m-4)=0$

or $m^2+6m+9-12m^2+44m+16=0$

or $11m^2-50m-25=0$

Factorising: $(m-5)(11m+5)=0$

Hence $m=5,-\dfrac{5}{11}$

Therefore roots will be equal if $m=5,-\dfrac{5}{11}$

Exercise 4.6

1. Find the nature of the roots of the given equations:
   - (i) $x^2-5x+6=0$
   - (ii) $9x^2+6x+1=0$
   - (iii) $x^2+x+1=0$
   - (iv) $5x^2+7x-1=0$
   - (v) $7x^2-2x+1=0$

2. For what value of 'm' the roots of the given equations are equal:

   (i) $x^2 + 3(m + 1)x + (4m + 5) = 0$

   (ii) $(m + 2)x^2 + (5m + 2)x + (4m + 1) = 0$

   (iii) $(m+5)x^2 + (3m - 1)x - (2m + 1) = 0$

3. The roots of the equation $(mx + c)^2 = 4ax$ will be equal if $c = \frac{a}{m}$

4. The roots of the equation $a^2(mx+c)^2 + b^2x^2 = a^2b^2$ will be equal if $c^2 = b^2 + a^2m^2$

5. Find 'h' if one root of $4x^2 + 7hx - h^2 + 9 = 0$ is zero.

6. Show that the roots of the equation.
   $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$
   are rational.

7. Show that the roots of the equation
   $(b - c)x^2 + (c - a)x + (a - b) = 0$
   are real.

8. Prove that the roots of the equation
   $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$
   are rational and they will be equal only if $a = b = c$

9. If the roots of
   $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$
   are equal, then $a^3 + b^3 + c^3 = 3abc$.

10. For what value of 'k' the expression
    $(4 - k)x^2 + 2(k + 2)x + 8k + 1$

will be a perfect square.

11. Show that the roots of the equation $(a+2b)x^2+2(a+b+c)x+(a+2c)=0$, are rational.

4.11 *Relations between the Roots and the Coefficients*

The sum of the roots α and β of the quadratic equation $ax^2+bx+c=0$ is

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2b}{2a}=-\frac{b}{a}=-\frac{\text{Coefficient of x}}{\text{Coefficient of x}^2}$$

The product of the roots α and β is

$$\alpha\ \beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{b^2-b^2+4ac}{4a^2}$$

$$=\frac{4ac}{4a^2}$$

$$=\frac{c}{a}=\frac{\text{constant}}{\text{Coefficient of x}^2}$$

Thus sum of roots $=-\frac{b}{a}$

Product of the roots $=\frac{c}{a}$

Example

If α, β are the roots of the equation $px^2 + qx + r = 0$, then find the values of (i) $\alpha^2 + \beta^2$ (ii) $(\alpha - \beta)^2$ (iii) $\alpha\beta^3 + \alpha^3\beta$

Solution

As α, β are the roots of $px^2 + qx + r = 0$

therefore $\alpha + \beta = -\dfrac{q}{p}$

$$\alpha\beta = \frac{r}{p}$$

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

So $\alpha^2 + \beta^2 = (\alpha+\beta^2) - 2\alpha\beta$

Substituting the values of $\alpha + \beta$ and $\alpha\beta$

(i) $$\alpha^2 + \beta^2 = \left(\frac{-q}{p}\right)^2 - 2\left(\frac{r}{p}\right)$$

$$\alpha^2 + \beta^2 = \frac{q^2}{p^2} - \frac{2r}{p} = \frac{q^2 - 2pr}{p^2}$$

(ii) $$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$= \left(\frac{-q}{p}\right)^2 - 4\left(\frac{r}{p}\right)$$

$$= \frac{q^2}{p^2} - \frac{4r}{p}$$

$$= \frac{q^2 - 4pr}{p^2}$$

(iii) $$\alpha\beta^3 + \alpha^3\beta = \alpha\beta(\alpha^2 + \beta^2)$$

$$= \alpha\beta\left[(\alpha + \beta)^2 - 2\alpha\beta\right]$$

$$= \frac{r}{p}\left[\left(\frac{-q}{p}\right)^2 - 2\left(\frac{r}{p}\right)\right]$$

$$= \frac{r}{p}\left[\frac{q^2 - 2pr}{p^2}\right]$$

$$= \frac{r\,(q^2 - 2pr)}{p^3}$$

4.11 Formation of a Quadratic Equation when its Roots are given

Let α, β be the roots of the desired equation.

Suppose $ax^2 + bx + c = 0$ be the equation with roots α and β.

Then $\alpha + \beta = -\frac{b}{a}$

and $\alpha\beta = \frac{c}{a}$

The equation can be written as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

the result obtained by dividing both members of the equation by '*a*'. Writing the last equation in terms of α and β as

$$x^2 - (\alpha + \beta)\,x + \alpha\,\beta = 0$$

or $x^2 - Sx + P = 0$

where S = sum of the roots and

P = product of the roots.

The formula for forming a quadratic equation with roots α and β is

$x^2 -$ (sum of the roots) $x +$ product of the roots $= 0$.

Example 1

Form a quadratic equation with roots 2 and 3

Solution

Roots of the required equation are 2 and 3

Sum of the roots $= 2 + 3 = 5$

Product of the roots $= (2)(3) = 6$

Desired equation is

$$x^2 - Sx + P = 0$$

or $$x^2 - 5x + 6 = 0$$

Second Method

$x = 2$ is equivalent to $x - 2 = 0$

and $x = 3$ is equivalent to $x - 3 = 0$

Hence $(x - 2)(x - 3) = 0$

or $$x^2 - 5x + 6 = 0$$

It is obvious that 2 and 3 are the roots of the equation.

Example 2

If α, β are the roots of $x^2 + px + q = 0$, find the equation with roots α^2 and β^2

Solution

As α, β are the roots of the equation

$$x^2 + px + q = 0$$

then $\alpha + \beta = -p$ ... (1)

$$\alpha.\beta = q \qquad (2)$$

Roots of the required equation are α^2 and β^2

Sum of the new roots $= \alpha^2 + \beta^2$

$$= (\alpha+\beta)^2 - 2\alpha\beta$$

Using (1) and (2)

$$\alpha^2 + \beta^2 = (-p)^2 - 2(q)$$

$$= p^2 - 2q$$

Product of the new roots $= \alpha^2\beta^2 = (\alpha\beta)^2$

Using (2)

$$(\alpha\beta)^2 = q^2$$

Thus desired equation is

$$x^2 - Sx + P = 0$$

or $\quad x^2 - (p^2 - 2q)x + q^2 = 0$

Exercise 4.7

1. Form a quadratic equation with the following given numbers as its roots

   (i) $2, -3$ (ii) $-1, 0$ (iii) $-3, 4$

   (iv) $3 + i, 3 - i$ (v) $2 \pm \sqrt{3}$

2. Without solving find the sum and the product of the roots of the following equations

   (i) $x^2 - x + 1 = 0$ (ii) $2x^2 + 5x - 1 = 0$

   (iii) $7x^2 - 5x + 4 = 0$ (iv) $3x^2 + 2x + 1 = 0$

   (v) $x^2 - 9 = 0$ (vi) $5x^2 + x - 7 = 0$

3. Find the value of 'k' given that:

(i) One root of $x^2 - (3k - 1)x + 4k = 0$ is 3

(ii) The sum of the roots of the equation

$$(2k - 1)x^2 + (4k - 1)x + k + 3 = 0 \text{ is } \frac{5}{2}$$

(iii) The product of the roots of the equation

$$(k + 1)x^2 + (4k + 3)x + k - 1 = 0 \text{ is } \frac{7}{2}$$

4. Find the quadratic equation with roots

(i) equal numerically but opposite in signs to those of the roots of $3x^2 + 5x - 7 = 0$

(ii) twice the roots of $5x^2 + 3x + 2 = 0$

(iii) more by '2' each than those of the roots of $4x^2 + 5x + 6 = 0$

5. If α, β are the roots of the equation $x^2 - 4x + 2 = 0$, find the equation whose roots are

(i) $2\alpha, 2\beta$ (ii) α^2, β^2 (iii) α^3, β^3

(iv) $\alpha + 2, \beta + 2$ (v) $\frac{1}{\alpha}, \frac{1}{\beta}$

(vi) $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$ (vii) $(\alpha - \beta)^2, (\alpha + \beta)^2$

6. If α, β are the roots of $px^2 + qx + r = 0$, find the equation with roots $(\alpha - \beta)^2, (\alpha + \beta)^2$

7. Find the condition that one root of $px^2 + qx + r = 0$ may be

(i) square of the other (ii) cube of the other

8. For what value of 'k', the sum of the roots of $3x^2 + kx + 5 = 0$ may be equal to the product of its roots

9. Form a quadratic equation if one of its roots is $2 + 3i$

10. Form a quadratic equation with one root $3 + \sqrt{2}$

11. If the differences between the corresponding roots of the equations $x^2 + bx + c = 0$ and $x^2 - cx + b = 0$ be the same then $b + c + 4 = 0$

12. The roots of the equation $px^2 + qx + q = 0$ are in the ratio $m : n$. Prove that $\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} + \sqrt{\frac{q}{p}} = 0$

13. Under what condition one root of $px^2 + qx + r = 0$ will be 'n' times the other.

4.12 Cube Roots of Unity

Let x be a cube root of unity

Then $x^3 = 1$

or $x^3 - 1 = 0$

Factorising : $(x - 1)(x^2 + x + 1) = 0$

Then either $x - 1 = 0$ or $x^2 + x + 1 = 0$

When $x - 1 = 0$, then $x = 1$

When $x^2 + x + 1 = 0$ then

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2} \text{ where } i = \sqrt{-1} \text{ or } i^2 = -1$$

Thus cube roots of unity are

$$1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$

The roots $\frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$ are called complex cube roots of unity as they have two parts one real and other imaginary. Let one of the complex roots be denoted by ω (omega)

$$\text{Suppose } \frac{-1 + i\sqrt{3}}{2} = \omega$$

$$\text{Then } \omega^2 = \left(\frac{-1 + i\sqrt{3}}{2}\right)^2$$

$$= \frac{1 - 3 - 2i\sqrt{3}}{2}$$

$$= \frac{-1 - i\sqrt{3}}{2}$$

= Other complex cube root of unity

$$\text{Similarly if } \omega = \frac{-1 - i\sqrt{3}}{2}$$

$$\text{Then } \omega^2 = \frac{-1 + i\sqrt{3}}{2}$$

Hence cube roots of unity are $1, \omega, \omega^2$

4.13 Properties of the Cube Roots of Unity

1. *The sum of the cube roots of unity is zero.*

$$\text{Here } 1+\omega+\omega^2 = 1+\frac{-1+i\sqrt{3}}{2}+\frac{-1-i\sqrt{3}}{2}$$

$$=\frac{2-1+i\sqrt{3}-1-i\sqrt{3}}{2}$$

$$=0$$

So $\quad 1+\omega+\omega^2=0$

2. *The product of the cube roots of unity is 1*

$$\text{Product of the cube roots } = (1)\cdot(\omega)\cdot(\omega^2)$$

$$=\omega^3$$

$$=1$$

3. *Each complex cube root of unity is reciprocal of the other,*

in other words

$$\omega=\frac{1}{\omega^2} \text{ and } \omega^2=\frac{1}{\omega}$$

As ω is a cube root of unity

therefore $\omega^3=1$

or $\omega\cdot\omega^2=1$

Hence $\omega^2=\dfrac{1}{\omega}$ and $\omega=\dfrac{1}{\omega^2}$

which prove the results.

4. *Powers of* ω.

As ω is a cube root of unity, therefore $\omega^3=1$

$$\omega^4 = \omega^3 \cdot \omega = \omega$$

$$\omega^5 = \omega^3 \cdot \omega^2 = \omega^2$$

$$\omega^6 = \left(\omega^3\right)^2 = (1)^2 = 1$$

The results can be extended to any integral powers of ω.

So that

$\omega^n = 1$: when n is a multiple of 3

$= \omega$: remainder is '1' when n is divided by 3

$= \omega^2$: remainder is '2' when n is divided by 3

Example 1

Evaluate ω^{46}, ω^{73} and ω^{92}

Solution

$$\text{(i) } \omega^{46} = \omega^{45} \cdot \omega = \left(\omega^3\right)^{15} \cdot \omega = (1)^{15} \cdot \omega = \omega$$

$$\text{(ii) } \omega^{73} = \omega^{72} \cdot \omega = \left(\omega^3\right)^{24} \cdot \omega = (1)^{24} \cdot \omega = \omega$$

$$\text{(iii) } \omega^{92} = \omega^{90} \cdot \omega^2 = \left(\omega^3\right)^{30} \cdot \omega^2 = (1)^{30} \cdot \omega^2 = \omega^2$$

Example 2

Prove that $(3 + \omega)(3 + \omega^2) = 7$

Solution

$$\begin{aligned} \text{L. H. S.} &= (3 + \omega)(3 + \omega^2) \\ &= 9 + (\omega + \omega^2)\, 3 + \omega^3 \\ &= 9 + (-1)\, 3 + 1 \quad \text{As } \omega + \omega^2 = -1 \text{ and } \omega^3 = 1 \\ &= 9 - 3 + 1 \end{aligned}$$

$$= 7 = \text{R. H. S}$$

Example 3

Evaluate $(-1 + i\sqrt{3})^5 + (-1 - i\sqrt{3})^5$

Solution

We have $\omega = \dfrac{-1 + i\sqrt{3}}{2}$

or $2\omega = -1 + i\sqrt{3}$

and $\omega^2 = \dfrac{-1 - i\sqrt{3}}{2}$

$$2\omega^2 = -1 - i\sqrt{3}$$

$$\begin{aligned} \text{Expression} &= (2\omega)^5 + \left(2\omega^2\right)^5 \\ &= 32\,\omega^5 + 32\omega^{10} \\ &= 32\left(\omega^2 + \omega\right) \\ &= 32\,(-1) \\ &= -32 \end{aligned}$$

Exercise 4.8

1. Find the cube roots of

(i) -1 (ii) -8 (iii) 27

2. If 'ω' is a cube root of unity then show that

(i) $(1 + 3\omega + \omega^2)(1 + \omega + 2\omega^2) = 2$

(ii) $(1 + 4\omega + \omega^2)(1 + \omega + 3\omega^2) = 6$

3. Evaluate

(i) $\left(1+\omega-\omega^2\right)^5+\left(3+\omega+3\omega^2\right)^5$

(ii) $\left(1+\omega-3\omega^2\right)^4+\left(5+\omega+5\omega^2\right)^4$

Prove that

4. $\left(\frac{-1+\sqrt{-3}}{2}\right)^{16}+\left(\frac{-1-\sqrt{-3}}{2}\right)^{16}=-1$

5. $(2+\omega)(2+\omega^2)=3$

6. $\left(1-\omega-\omega^2\right)^5=32$

7. $\omega^{28}+\omega^{38}=-1$

8. Show that the product of $2n$ factors

$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$...... is 1

9. Show that the complex cube roots of unity are reciprocals of each other.

10. Show that the product of the cube roots of unity is one.

Prove that

11. $x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$

12. $x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y)$

4.14 Solution of Problems Involving Equations

Equations are largly used for the solutions of various kinds of problems. The roots of the equations provide answer to these problems.

While solving the problems, keep the following points in mind:

1. Read carefully the problem involved and understand the relations between the quantities stated therein.
2. Represent the unknown quantities in symbols x, y, z etc.
3. State the conditions given in the problems in symbolic language so as to obtain an equation or equations.
4. Solve the equation thus obtained for unknown quantities and verify the results.

Methods for solving such problems are explained in the following examples:

Example 1

Find two consecutive positive integers whose product is 56

Solution

Let the two consecutive integers be x and $x + 1$

Then according to the given condition

$$x(x + 1) = 56$$

or $$x^2 + x - 56 = 0$$

or $$(x + 8)(x - 7) = 0$$

Solving : $x = -8, 7$

Required integers are $x = 7$ and $7 + 1 = 8$. We neglect the negative integers.

Example 2

The area of a rectangle is 320 square meters. Its one side is 4 meter larger than the other side. Find its sides.

Solution

Let one of the sides be x. Then the other side is $x + 4$. Thus area is $x(x + 4)$.

According to the given conditon

$$x(x + 4) = 320$$

or $$x^2 + 4x = 320$$

$$x^2 + 4x - 320 = 0$$

Factorising: $(x + 20)(x - 16) = 0$

Hence $x = 16, -20$

Neglecting negative value of x, the sides are 16 and 20 meters.

Example 3

The sum of two numbers is 9 and the sum of their squares is 41. Find the numbers.

Solution

Let the numbers be x and y

Therefore, by the given conditions

$$x + y = 9 \quad \dots \quad (1)$$

$$x^2 + y^2 = 41 \quad \dots \quad (2)$$

From (1) $x = 9 - y \quad \dots \quad (3)$

Putting this value of x in (2)

$(9 - y)^2 + y^2 = 41$

Simplifying to get

$y^2 - 9y + 20 = 0$

Factorising : $(y - 4)(y - 5) = 0$

If $y - 4 = 0$, then $y = 4$

If $y - 5 = 0$, then $y = 5$

when $y = 4$ then $x = 5$

when $y = 5$, then $x = 4$

Hence the numbers are 4 and 5

Example 4

The sum of the coordinates of a point is 7 and the sum of their squares is 25. Find the coordinates of the point.

Solution

Let the coordinates of the point be (x, y)

By the given condition

$x + y = 7 \quad \ldots \quad (1)$

and $x^2 + y^2 = 25 \quad \ldots \quad (2)$

From (1) $x = 7 - y \quad \ldots \quad (3)$

Putting this value of x in (2)

$(7 - y)^2 + y^2 = 25$

or $49 - 14y + y^2 + y^2 = 25$

$2y^2 - 14y + 24 = 0.$

or $\quad y^2 - 7y + 12 = 0$

or $\quad (y - 3)(y - 4) = 0$

The equation gives $y = 3$ and $y = 4$

When $y = 3$, then $x = 4$

When $y = 4$, then $x = 3$

Thus the coordinates are (4, 3) or (3, 4)

Exercise 4.9

1. A line segment 48 meters is divided into two parts and one part is 4 meters longer than the other. Find the length of the each part.
2. At what rate will Rs:750/- be the profit for five years on Rs:6000/-?
3. What percentage of 20% of solution of hydrocholoric acid would be drawn off and replaced by water to give 15% solution?
4. Square of a number added to the number gives 6, find the number.
5. Find the sides of a rectangle with area 128 m^2 if its length is twice its breadth.
6. Find two positive consecutive integers whose squares differ by 27.
7. Find three consecutive positive integers such that the sum of their squares is equal to 110.
8. The sum of the squares of the two digits of a positive integral number is 65 and the number is 9 times the sum of its digits, find the numbers.

9. The sum of the coordinates of a point is 8 and the sum of their squares is 34, find them.

10. A number added to 42 results in the square of the number. Find the number.

11. The sum of two digits of a number is 8. A number is increased by 18, when the digits are reversed, find the number.

12. The number of two digits is decreased by 27, when the digits are reversed. The sum of the digits being 9, find the number.

13. The sum of the ages of a father and a son is 56 years. Eight years ago, the father was 3 times as old as his son. How old is the son now?

14. The half sum of two numbers is 20. One quarter of their difference is 1/2. Find the numbers.

15. A number consisting of three digits has hundred digit four times the unit digit. The sum of the digits is 10. If the digits are written in the reversed order the number is decreased by 297. Find the number.

16. The sum of the two numbers is 32 and their product is 240, find the numbers.

17. Two numbers differ by 7 and their product is 120, find the numbers.

18. The length of a rectangle is 4 cm more than its breadth, the area of a rectangle is 96 cm^2. Find its

sides.

19. Two numbers differ by 5, the difference of their squares is 95. Find the numbers.

20. The lengths of the sides of a right angled triangle are $2x + 2$, $2x + 3$ and x centimeters. Find the lengths of the sides.

21. The perimeter of a rectangle is 50 meters and its area is 150 m^2. Find the sides of the rectangle.

22. How long will Akram and Aslam take to do a job working together, if Akram alone can do that job in 5 days and Aslam alone takes 8 days to do that job alone.

23. The sum of two numbers is 5. The square of one added to twice the square of the other yields 17. Find the numbers.

24. The product of two numbers is one. Two added to the five times of one number yields twice the second number. Find the numbers.

25. Eight added to the square of a number results in 6 times the number. Find the number.

26. The square of a number added to twice the number results in 28. Find the number.

27. Three times the square of a number decreased by 12 results in 5 times the number. Find the number.

28. A number added to twice its reciprocal yields $\frac{11}{3}$. Find the number.

29. Three times the square of a number decreased by 12 times the number results in 135. Find the number.

30. A number added to its reciprocal yields $\frac{13}{6}$. Find the number.

Chapter 5

Sequences and Series

5.1 Sequences

'Sequence' is a word of common language. It denotes certain objects or events occuring in some order. For instance, the periods that a college student has to attend on a day of the week form a sequence. In mathematics we have to deal with number sequences. In order to give this term a precise meaning we take help of the following examples.

Consider the following arrays of numbers which are arranged in order and have been formed by some rule. (Find out the rule in each case yourselves):-

(i) 2, 4, 6, 8, 10, 12, 14, ...

(ii) 3, 12, 48, 192, ...

(iii) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15} \cdots$

These are a few examples of sequences. Thus we can say that:-

**A sequence is an arrangement of numbers formed according to some definite rule.*

In a sequence of numbers the relative position of each number is specifically designated as first, second, third and so on. Each number in the sequence is called a *term*. Not only the value of each term is important but also so is the relative position of the term among the other terms. A sequence is also called a *progression*.

The *nth term* or *general term* of a sequence is sometimes denoted by $s(n)$. nth term is also denoted by a_n. In this way a_1, a_2, a_3, a_4 etc. will be the first, second, third, fourth terms

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* We have introduced 'sequences' in an informal or intuitive manner. Formally, a sequence is defined with the help of a sequence function.

Def. A *sequence function* is a function whose domain is N, the set of natural numbers.

For example, the function defined by

$$s(n) = n^2, n \in N$$

is a sequence function.

The range elements of a sequence function considered in the order

$$s(1), s(2), s(3), ...$$

are said to form a number sequence or simply a *sequence*. Unless we take the function with domain N or a subset of N and the range as a set of real numbers, we cannot call it a number sequence.

In the above example the sequence arising from the function is

$$1^2, 2^2, 3^2, 4^2, ...$$

or $1, 4, 9, 16, ...$

A sequence may be *finite* or *infinite* according as the numberr of its terms is finite or infinite.

etc of the sequence.

A few more examples of sequences are given below:

General term (a_n)	***Sequence***
(1) $2n$	$2, 4, 6, 8, \ldots$
(2) n^2	$1, 4, 9, 16, \ldots$
(3) $\frac{1}{n}$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
(4) $(-1)^n$	$-1, 1, -1, 1, \ldots$
(5) $\frac{n+1}{n}$	$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \ldots$
(6) $a+(n-1)d$	$a, a+d, a+2d, \ldots$
(7) $a\,r^{n-1}$	$a, ar, ar^2, ar^3, \ldots$

Exercise 5.1

(a) Write down the first four terms of each of the sequences whose general terms are as given below:-

1. $n-5$ 2. $\frac{n^2-2}{2}$ 3. $1+\frac{2}{n}$

4. $\frac{n\,(n-1)}{2}$ 5. $(-1)^{n+1}$ 6. $\frac{(-1)^n\,(n-2)}{n}$

7. $4+3\,(n-1)$ 8. $2 \cdot 1\,(-3)^{n-1}$

9. $\frac{1}{a+(n-1)\,d}$

(b) Write down the next two terms of each sequence:-

10. $4, 8, 12, 16, \ldots$ 11. $2, 5, 8, 11, \ldots$

12. $-1, 1, -1, 1, \ldots$ 13. $x^2, x^3, x^4, x^5, \ldots$

14. $-x, x^3, -x^5, x^7, \ldots$

5.2 Arithmetic Sequence

Consider the sequence of numbers

2, 4, 6, 8, 10, 12, 14, ...

Each term after the first term is obtained by adding a fixed number 2 to the preceding term. This sequence is an example of arithmetic sequence or progression. Thus

Definition:

A sequence every term of which after the first is obtained from the preceding term by adding a fixed number, is called an *arithmetic sequence* or *arithmetic progression* (abbreviated as A.P.).

The constant difference between any two consecutive terms of an A.P. is called its *common difference* (c.d.)

In the foregoing example,

first term = 2, common difference (c.d.) = 2

Example 1

4, 9, 14, 19, 24, ... is an A.P.

First term = 4, c.d. = 9 – 4 = 14 – 9 = 5

Example 2

Consider the A.P. 10, 7, 4, 1, ...

First term = 10, second term = 7

c.d. = – 3

Example 3

The sequence

40, 45, 50, 55, ... is an A.P

First term = 40, second term = 45

c.d. = 5

5.3 General Term of an Arithmetic Progression

The standard or general form of an A.P. is

$$a, a + d, a + 2d, a + 3d, \ldots$$

Its first term = a, c.d. = d

In this case

$$a_1 = a = a + (1 - 1)\, d$$

$$a_2 = a + d = a + (2 - 1)\, d$$

$$a_3 = a + 2d = a + (3 - 1)\, d$$

$$\therefore\ a_n = a + (n - 1)\, d$$

Example 4

Find the first four terms of the A.P. in which $a = 7$, $d = -4$ (d stands for common difference). Also write down its general term and 40th term.

Solution

$$a_1 = 7, d = -4$$

First term = $a_1 = 7$

$$a_2 = a + d = 7 + (-4) = 3$$

$$a_3 = a + 2d = 7 + 2(-4) = 7 - 8 = -1$$

$$a_4 = a + 3d = 7 + 3(-4) = 7 - 12 = -5$$

$\therefore$ the first four terms are 7, 3, – 1, – 5

$$\text{General term} = a_n = a + (n-1)d = 7 + (n-1)(-4)$$
$$= 7 - 4n + 4$$
$$= 11 - 4n$$
$$a_{40} = \text{40th term} = a + (40-1)d = 7 + 39(-4)$$
$$= 7 - 156 = -149$$

Example 5

Find the number of terms in an A.P. in which

$$l^* = a_n = 130, d = 25, a = 5$$

Solution

$$l = a_n = 130, d = 25, a = 5$$
$$a_n = a + (n-1)d$$
$$130 = 5 + (n-1)25$$
$$130 = 5 + 25n - 25$$
$$\text{or } 130 + 20 = 25n \qquad \therefore n = \frac{150}{25} = 6$$

Example 6

If the 12th term of an A.P. is 19 and 17th term is 29, find the first term and the common difference.

Solution

Let a be the first term and d the common difference of the sequence.

$$\text{12th term} = a_{12} = a + (12-1)d$$

---

* 'l' denotes the 'last term' of a finite A.P.

or $19 = a + 11d.$ ... (1)

17th term $= a_{17} = a + (17 - 1)d$

or $29 = a + 16d$ ... (2)

Subtracting (1) from (2) we get

$$5d = 10 \qquad \therefore\ d = \frac{10}{5} = 2$$

Substituting $d = 2$ in (1) we get,

$$a + 22 = 19 \qquad \therefore\ a = 19 - 22 = -3$$

Hence first term $= -3$, common difference $= 2$

Exercise 5.2

(a) Write the next two terms in each of the following A.P's:-

1. $5, 9, 13, \ldots$ 2. $3, 7, 11, \ldots$

3. $-1, -5, -9, \ldots$ 4. $x, x+1, x+2, \ldots$

5. $x + a, x + 3a, x + 5a, \ldots$

6. $2x + 1, 2x + 4, 2x + 7, \ldots$

7. $x, 2x, 3x, \ldots$

(b) Find the indicated term of the A.P.:-

8. a_7 of $7, 11, 15, \ldots$ 9. a_{12} of $2, \frac{5}{2}, 3, \ldots$

10. a_{10} of $\frac{3}{4}, 2, \frac{13}{4}, \ldots$

11. Find the 20th term of the A.P. whose 3rd term is 7 and 8th term is 17.

12. If the 5th term of an A.P. is -16 and the 20th term is -46, what is its 12th term?

13. Which term of the A.P. 4, 1, – 2, ... , is – 77 ?

14. Which term of the sequence 3, 9, 15, ... , is 117 ?

(c) In each of the following data one of a, d, n and a_n is missing. Find the missing number:-

15. $a = 3, a_n = 19, n = 5$

16. $a = 4, d = -2, a_n = -12$

17. $d = 4, a_n = 40, n = 12$

18. $a = 30, n = 20, d = -4$

19. Find the sequence whose general term is

(i) $4n + 1$ (ii) $3n - 1$

Verify that each sequence is an A.P.

20. If l, m, n, be the pth, qth and rth terms of an A.P., show that

(i) $p(m - n) + q(n - l) + r(l - m) = 0$;

(ii) $l(q - r) + m(r - p) + n(p - q) = 0$.

5.4 Arithmetic Means

A number 'A' is said to be the arithmetic mean between two numbers a and b if a, A, b form an A.P.

In such a case a, A, b are in A.P.,

$$\therefore A - a = b - A$$

$$\text{Hence } A = \frac{a+b}{2}$$

'Arithmetic Mean' is abbreviated as 'A.M.'

Example 1

Find the A.M. of $\frac{1}{4}$ and $\frac{5}{8}$

Solution

Here $a = \frac{1}{4}$, $b = \frac{5}{8}$

$$\therefore \text{reqd. A.M.} = \frac{\frac{1}{4}+\frac{5}{8}}{2} = \frac{2+5}{8\times 2} = \frac{7}{16}$$

Definition

$A_1, A_2, A_3, \ldots, A_n$ are said to be n arithmetic means between a and b if $a, A_1, A_2, A_3, \ldots, A_n, b$ form an A.P.

To Insert n Arithmetic Means (A.M's) between given numbers a and b.

Let $A_1, A_2, A_3, \ldots, A_n$ be n A.M's between a and b. Then

$a, A_1, A_2, A_3, \ldots, A_n, b$

is a finite A.P. having $n + 2$ terms.

Suppose d is the c.d. of the A.P. Then

$$A_1 = a + d$$

$$A_2 = a + 2d$$

$$\ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots$$

$$A_n = a + (n)d$$

$$b = a + (n + 1)d$$

From the last equation, $d = \frac{b-a}{n+1}$

$$\therefore A_1 = a + \frac{b-a}{n+1},$$

$$A_2 = a + \frac{2(b-a)}{n+1}$$

$$A_3 = a + \frac{3(b-a)}{n+1}$$

... ...

... ...

$$A_n = a + \frac{n(b-a)}{n+1},$$

Example 2

Insert 5 A.M's between 3 and 12

Solution

Let A_1, A_2, A_3, A_4 and A_5 be the five arithmetic means between 3 and 12. Then 3, A_1, A_2, A_3, A_4, A_5, 12 are in A.P.

Here $a = 3$. Let d be the c.d.

12 is the 7th term so that

$$12 = a + 6d = 3 + 6d$$

$$\text{or} \quad 6d = 9 \quad \text{Therefore } d = \frac{9}{6} = \frac{3}{2}$$

$$\therefore A_1 = \text{2nd term} = a + d = 3 + \frac{3}{2} = \frac{9}{2}$$

$$A_2 = \frac{12}{2} = 6, \quad A_3 = \frac{15}{2},$$

$$A_4 = 9, \quad A_5 = \frac{21}{2},$$

Thus the five A.M's between 3 and 12 are

$$\frac{9}{2}, 6, \frac{15}{2}, 9, \frac{21}{2}$$

Note: $12 = 7\text{th term} = a + 6d = 3 + 6 \times \frac{3}{2} = 3 + 9 = 12$

Hence the procedure is correct.

Exercise 5.3

Find the A.M. between.

1. 4 and 1000
2. -5 and 40
3. 18 and 54
4. $2 + \sqrt{3}$ and $2 - \sqrt{3}$
5. $\sqrt{5} - 4$ and $\sqrt{5} + 4$
6. 84 and 107
7. Insert two A.M's between -5 and 40
8. Insert six A.M's between 12 and -9
9. Insert four A.M's between $\frac{1}{2}$ and 8
10. Insert three A.M's between -18 and 4
11. Find n so that $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the A.M. between a and b.

5.5 Series

A series is defined to be the sum of the terms of a sequence.

For example, for any natural number n, with the finite sequence

$$4, 7, 10, \ldots, (3n + 1),$$

there is associated the finite series

$4 + 7 + 10 + ... + (3n + 1)$

We denote this sum by S_n so that the above sum can be written as:

$S_n = 4 + 7 + 10 + ... + (3n + 1)$

The sub-script n in S_n indicates that the number of terms to be added is n.

Similarly, with the finite sequence

$2, 4, 8, 16, 32, ..., 2^n$

there is associated the *finite series*

$S_n = 2 + 4 + 8 + 16 + 32 + ... + 2^n$.

Since the terms in the series are the same as those in the sequence, we can refer to the first, second or general term of the series in the same manner as we do for a sequence.

Finite and Infinite Series

A series may be finite as we saw in the above two examples and a series may be infinite as for example,

$S = 4 + 7 + 10 + ...$

Similarly the series,

$S = 3 + 9 + 27 + 81 + ...$

is also an infinite series.

5.6 Arithmetic Series

The arithmetic series

$S_n = a + (a + d) + (a + 2d) + ... + [a + (n - 1)d]$

is an arithmetic series in the general form or standard form.

The Sum of an A.S.[*]

The sum of n terms of the arithmetic series is obtained as follows:

Let S_n denote the sum of first n terms of the arithmetic series (A.S) in the standard form, then

$$S_n = a + (a+d) + (a+2d) + \dots + [a+(n-1)d] \quad \dots \text{(I)}$$

Also writing the terms in the reverse order, S_n can be written

$$S_n = [a+(n-1)d] + [a+(n-2)d] + \dots + (a+d) + a \quad \dots \text{(II)}$$

Adding (I) and (II) term by term,

$$2S_n = [2a+(n-1)d] + [2a+(n-1)d] + \dots + [2a+(n-1)d] \dots (n \text{ terms})$$

$$= n[2a+(n-1)d]$$

$$\text{Hence} \quad S_n = \frac{n}{2}[\,2a+(n-1)d] \quad \dots \quad \text{(III)}$$

$$\text{Also} \quad S_n = \frac{n}{2}[a + a + \overline{n-1}\,d]$$

$$\therefore \quad S_n = \frac{n}{2}[a+l]$$

Where l is nth term or last term.

Example 1

Find S_n for the series $4 + 7 + 10 + \dots$

Solution

$S_n = 4 + 7 + 10 + \dots$ to n terms

Here $a = 4$, $d = 7 - 4 = 3$,

---

[*] The "sum of an A.S." is sometimes loosely called 'sum of an A.P.'

Therefore, $S_n = \frac{n}{2}[2a + (n-1)d]$

$$= \frac{n}{2}[2 \times 4 + (n-1)3]$$

$$= \frac{n}{2}[8 + 3n - 3]$$

$$= \frac{n}{2}(3n + 5)$$

$$= \frac{n(3n+5)}{2}.$$

Example 2

Find the sum of all the integral multiples of 7 between 8 and 110.

Solution

We have to find the sum

$$14 + 21 + 28 + \ldots + 105$$

$$= 7(2 + 3 + 4 + \ldots + 15)$$

$$= 7 \times \frac{14}{2}(2 + 15)$$

[There are 14 terms in the series so that $n = 14$. Also $a = 2$, $l = 15$],

$$= 7 \times 7 \times 17 = 833$$

Example 3

Find the sum of the series

$$3 + 11 + 19 + \ldots \text{ to 16 terms.}$$

Solution

$$a = 3, d = 11 - 3 = 8, n = 16$$

$$S_n = S_{16} = \frac{16}{2}[2 \times 3 + (16 - 1) \times 8]$$

$$= 8(6 + 120)$$

$$= 8 \times 126$$

$$= 1008$$

Exercise 5.4

Find the sum of each of the following series:-

1. $5 + 8 + 11 + 14 + \dots$ to n terms.
2. $2 + 7 + 12 + \dots$ to 12 terms.
3. $7 + 16 + \dots + 88$
4. $-1 + 1 + 3 + \dots + 21$
5. $(p+q) + (2p+q) + (3p+q) + \dots + (8p+q)$

   [*Hint:* $a = p + q, d = p, n = 8$]
6. $(l - m) + (l - 2m) + (l - 3m) + \dots + (l - 12m)$.
7. $\frac{2}{3} + 1 + \frac{4}{3} + \frac{5}{3} + \dots$ to 17 terms.
8. $5 + 3 + 1 - 1 - \dots$ to 10 terms.
9. Sum the series: $1 + 3 - 5 + 7 + 9 - 11 + 13 + 15 - 17 + \dots$ to $3n$ terms

[*Hint*: Express the series as sum of 3 series, each containing n terms]

10. Obtain the sum of all the integers in the first 1000

positive integers which are neither divisible by 5 nor by 2.

[*Hints*: Find the sums:

$S_1 = 1 + 2 + 3 + 4 + 5 + ... + 1000$

$S_2 = 2 + 4 + 6 + ... + 1000$ (500 terms: integers disivible by 2)

$S_3 = 5 + 15 + 25 + ... + 995$ (200 terms: integers divisible by 5)

The required sum = $S_1 - S_2 - S_3$.

Or find the sum of the series

$1 + 3 + 7 + 9 + 11 + 13 + 17 + 19 + ... + 991 + 993 + 997 + 999$

by splitting it into 4 series:

$1 + 11 + 21 + ... + 991$

$3 + 13 + 23 + ... + 993$

$7 + 17 + 27 + ... + 997$

$9 + 19 + 29 + ... + 999$]

11. The nth term of a series is $4n + 1$. Find its sum. Find the sum of its 100 terms.

12. The sum of three numbers in A.P. is 24 and their product is 440. Find the numbers.

[*Hint*: Suppose the numbers are $a - d, a, a + d$]

13. Find four numbers in A.P. whose sum is 24 and the sum of whose squares is 164.

[*Hint*: Suppose the numbers are $a - 3d, a - d, a + d, a + 3d$]

14. Find the five numbers in A.P. whose sum is 30 and the sum of whose squares is 190

5.7 Geometric Sequence (G.S.)

A sequence of numbers in which every term after the first is obtained from the preceeding term by multiplying it with a constant number is called a geometric sequence or geometric progression, shortly written as G.P. The constant ratio of any two consecutive terms of a G.P. is called its *common ratio*, which is generally denoted by r.

The constant ratio of a G.P. may be any number not equal to 1.

A few examples of a G.P. are:

(1) 2, 4, 8, ... (common ratio = 2).

(2) – 2, 4, – 8,... (common ratio = – 2)

(3) $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \dots$ (common ratio = $\frac{1}{2}$)

(4) 0.1, 0.01, 0.001,... (common ratio = 0.1 = $\frac{1}{10}$)

The *standard form* or *general form* of a G.P. is

$a, ar, ar^2, \dots$

Its first term is 'a' and common ratio is r.

Example 1

Write the next four terms of the sequence: 3, 6, 12, ...

Solution

Give u G.P. is

3, 6, 12, ...

First term = 3. Second term = 6

Common ratio = 6 + 3 = 2.

$\therefore$ next four terms are: 24, 48, 96, 192.

5.8 General Term of a G.P

For the G.P. in the standard form,

$$a_1 = a = a\,r^0 = a\,r^{1-1}$$
$$a_2 = a\,r = a\,r^{2-1}$$
$$a_3 = a\,r^2 = a\,r^{3-1}$$
$$a_4 = a\,r^3 = a\,r^{4-1}$$

Hence $a_n = a\,r^{n-1}$

Example 2

Write the sequence in which

$a = 5, r = 3, n = 8$

Solution

First term = $a_1 = a = 5$,

$r = 3$

n = number of terms = 8

a_n will be a_8 in this case.

$a_1 = a = 5,\ a_2 = 5 \times 3 = 15$

$a_3 = 15 \times 3 = 45,\ a_4 = 45 \times 3 = 135$

$a_5 = 135 \times 3 = 405,\ a_6 = 405 \times 3 = 1215$

$$a_7 = 1215 \times 3 = 3645,\ a_8 = 3645 \times 3 = 10935$$

Therefore, the required sequence is as follows:

5, 15, 45, 135, 405, 1215, 3645, 10935.

Example 3

Write the G.P. (with positive terms) whose second term is 9, 4th terms is 1 and $n = 6$.

Solution

Second term $= a_2 = a\,r = 9$ ... (1)

4th term $= a_4 = a\,r^3 = 1$ ... (2)

Dividing (2) by (1) we get,

$r^2 = \frac{1}{9} \Rightarrow r = \frac{1}{3}$ (rejecting its negative value)

Substituting $r = \frac{1}{3}$ in (1), we get,

$a \times \frac{1}{3} = 9 \qquad \therefore\ a = 3 \times 9 = 27$

Hence the first term = 27, common ratio $= \frac{1}{3}$

so that the required G.P. is

$27, 9, 3, 1, \frac{1}{3}, \frac{1}{9}$ (because $n = 6$)

Example 4

Find the first term of the G.P. whose 5th term is 48 and common ratio is 2

Solution

If a is the first term:

$$a_5 = a\,r^4 = 48$$

$$\Rightarrow a \times 16 = 48 \Rightarrow a = \frac{48}{16} = 3.$$

Example 5

The population of a town increases at the rate of 10% annually. Its present population is 200000. What will be its population at the end of 5 years?

Solution

If the present population be denoted by a, the population at the end of the 1st year, 2nd year, (110% of $a = a \times 1.1$)

3rd year, ... will be

$$a\,(1.1), a(1.1)^2, a(1.1)^3, \ldots$$

Therefore population at the end of 5 years

$$= a\,(1.1)^5$$

$$= 200000 \times 1.61051$$

$$\cong 322102$$

Exercise 5.5

(a) Write the next five terms of the following G.Ps.

1. 2, 10,... 2. 27, 9, 3,... 3. $1, \frac{1}{2}, \frac{1}{4}, \ldots$

(b) Find the nth term of the G.P.

4. $a = 8, r = \frac{3}{2}, n = 5.$ 5. $a = -1, r = -4, n = 6.$

6. $a = 3, r = -2, n = 10$ 7. $a = \frac{125}{8}, r = \frac{2}{5}, n = 8$

(c) Write down the finite geometric sequence which satisfies the given conditions (n is the total number of terms):-

8. $a = 3, r = 5, n = 6$ 9. $a = 128, r = -\frac{1}{2}, n = 8.$

10. First term = 2, second term = -6, $n = 5$.

11. Third term = 9, 6th term = $\frac{1}{3}$, $n = 8$.

12. If the second term of a G.P. is 2 and the 11th term is $\frac{1}{256}$, what is the first term? What is the nth term?

Find one of a, r, n and l whichever is missing:-

13. $l = a_n = 192, n = 7, r = 2.$

14. $a = 4, l = a_n = \frac{1}{8}, r = \frac{1}{2}$

15. $a = 4096, l = a_n = -4, n = 6.$

16. $a = 4, r = \frac{1}{2}, n = 5.$

17. The value of an automobile depreciates at the rate of 15% per year. What will be the value of an automobile 3 years hence which is now purchased for Rs.45000?

18. The population of a town increases at the rate of 15% annually. If its present population is 150000,

what will it become after 4 years?

19. A rubber ball is dropped from a height of 16 dm. It continuously rebounds to $\frac{3}{4}$ of the distance of its previous fall. How high does it rebound the fourth time?

5.9 Geometric Means

A number G is said to be the *geometric mean* (G.M.) between two numbers *a* and *b* if *a, G, b* are in a G.P.

In such a case *a, G, b* are in a G.P.

$$\Rightarrow \frac{G}{a} = \frac{b}{G}$$

$$\Rightarrow G^2 = ab$$

Hence $G = \pm\sqrt{a\,b}$

Example 1

Find the G.M. between 2 and 8.

Solution

Let G be the G.M, then

$$G = \pm\sqrt{2 \times 8} = \pm\sqrt{16} = \pm 4$$

Hence $G = \pm 4$

$G_1, G_2, G_3, \ldots, G_n$ are called *n geometric means* (G.M's.) between two numbers *a* and *b* if

$$a, G_1, G_2, G_3 \ldots, G_n, b$$

form a finite G.P

To Insert n G.M's. between two Numbers a and b

Let $G_1, G_2, G_3, \ldots, G_n$ be n G.M's between a and b. Then

$a, G_1, G_2, G_3, \ldots, G_n, b$ are in G.P.

$\therefore \quad G_1 = a\,r$ (where r is the c.r. of the G.P.)

$G_2 = a\,r^2$

... ...

$G_n = a\,r^n, b = a\,r^{n+1}$

Hence $r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

and the required G.Ms are:

$$a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}, \ldots, a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Note: The next term after the nth G.M. is

$a\left(\dfrac{b}{a}\right)^{\frac{n+1}{n+1}} = a \cdot \dfrac{b}{a} = b$. This shows the correctness of the procedure.

Example 2

Find three G.Ms between $\dfrac{27}{8}$ and $\dfrac{2}{3}$

Solution

Let G_1, G_2, G_3 be the three geometric means between $\dfrac{27}{8}$ and $\dfrac{2}{3}$

$\frac{27}{8}, G_1, G_2, G_3, \frac{2}{3}$ are in a finite G.P.

$a = \frac{27}{8}$. So $\frac{2}{3}$ is the 5th term of the G.P.

$$\therefore \quad \frac{2}{3} = a\,r^{n-1} = a\,r^4 = \frac{27}{8} \times r^4$$

$$\text{or } r^4 = \frac{2}{3} \times \frac{8}{27} = \frac{16}{81}$$

$$\Rightarrow r = \pm\frac{2}{3}$$

If $r = \frac{2}{3}$

$$G_1 = a\,r = \frac{27}{8} \times \frac{2}{3} = \frac{9}{4},$$

$$G_2 = a\,r^2 = \frac{27}{8} \times \frac{4}{9} = \frac{3}{2},$$

$$G_3 = a\,r^3 = \frac{27}{8} \times \frac{8}{27} = 1.$$

If $r = -\frac{2}{3}$

$$G_1 = a\,r = \frac{27}{8} \times \frac{-2}{3} = \frac{-9}{4},$$

$$G_2 = a\,r^2 = \frac{27}{8} \times \left(\frac{-2}{3}\right)^2 = \frac{3}{2},$$

$$G_3 = a\,r^3 = \frac{27}{8} \times \left(\frac{-2}{3}\right)^3 = -1$$

Example 3

Insert 6 G.M's between 2 and 256.

Solution

Let the 6 G.M's be $G_1, G_2, G_3, G_4, G_5, G_6$.

Then 2, $G_1, G_2, G_3, G_4, G_5, G_6$, 256 are in a G.P.

If r is the c.r., then

$256 = 8\text{th term} = 2 \,.\, r^7$

$\therefore\ r = 2$

(128 has only one real 7th root)

Hence required G.M's are

2.2, 2.4, 2.8, 2.16, 2.32, 2.64

or 4, 8, 16, 32, 64, 128.

Note: If the number of required G.M's is even, there is only one set of G.M's, but if the number of required G.M's is odd, two sets of G.M's are obtained as in the last but one example.

Exercise 5.6

Find the G.M between:

1. 4, 64 2. $\frac{1}{3}$, 243 3. $\frac{8}{9}, \frac{9}{8}$

4. Insert three G.M's between 256 and 1.

5. Insert four G.M's between 2 and – 486.

6. Insert three G.M's between $\frac{1}{2}$ and $\frac{1}{162}$.

7. For what value of n is $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ the G.M between a and b, where a and b are not zero simul-

taneously.

5.10. Geometric Series

Associated with the G.P.

$a, ar, ar^2, \ldots, ar^{n-1} \ldots$ is the geometric series

$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$

Sum of n Terms of a Geometric Series

Let S_n be the required sum i.e.

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad \ldots \quad \text{(i)}$$

Multiplying each side by r we get

$$r\,S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \quad \text{(ii)}$$

Subtracting (ii) from (i) we get

$$S_n - rS_n = a - ar^n$$

or $\quad S_n(1 - r) = a(1 - r^n)$:

$$\therefore \quad S_n = \frac{a(1 - r^n)}{1 - r} \quad \ldots\text{(A)} \qquad (r \neq 1)$$

Alternative form of S_n:

$$S_n = \frac{a - ar^n}{1 - r} = \frac{a - ar^{n-1} \cdot r}{1 - r}$$

$$\text{i.e } S_n = \frac{a - lr}{1 - r} \quad \ldots \quad \text{(B)}$$

where $l = ar^{n-1}$ = last term

Note: The latter formula can be used when the last term of the geometric series besides a and r is given:

Note 2: Formula (A) may be written in the form

$$S_n = \frac{a\,(r^n - 1)}{r - 1} \quad \dots \qquad \text{(C)}$$

Form (A) or (C) may be used according as

$|r| < 1$ or $|r| > 1$

Also the form (B) or the form

$$S_n = \frac{lr - a}{r - 1} \quad \dots \qquad \text{(D)}$$

may be used according as $|r| < 1$ or $|r| > 1$

Example 1

Find S_n for the series $2 + 4 + 8 + \dots + 2^n$

Solution

Here $a = 2, r = \frac{4}{2} = 2, n = n.$

$$\therefore S_n = \frac{a\,(r^n - 1)}{r - 1} \quad \text{[using formula (c)]}$$

$$S_n = \frac{2\,(2^n - 1)}{2 - 1}$$

$$S_n = 2^{n+1} - 2$$

Example 2

Find the sum of the series

$\frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \dots$ upto 8 terms.

Solution

Here $a = \frac{2}{3}, r = \frac{1}{3} \div \frac{2}{3} = \frac{1}{3} \times \frac{3}{2} = \frac{1}{2}$

$$\text{Therefore } S_n = S_8 = \frac{\frac{2}{3}\left[1-\left(\frac{1}{2}\right)^8\right]}{1-\frac{1}{2}}$$

$$= \frac{\frac{2}{3}\left[1-\frac{1}{256}\right]}{\frac{1}{2}}$$

$$= \frac{2}{3} \times \frac{2}{1} \times \frac{255}{256} = \frac{85}{64}$$

Example 3

Find the sum of the geometric series

$3 - 6 + 12 - \ldots - 1536$

Solution

Here $a = 3, r = -6 \div 3 = 12 \div (-6) = -2$, $l = -1536$.

$$\text{Therefore, } S_n = \frac{lr - a}{r - 1} = \frac{(-1536)\,(-2)\ -3}{-2-1}$$

$$= \frac{3069}{-3} = -1023$$

Exercise 5.7

Find the sum of each of the following series:-

1. $3 + 3^2 + 3^3 + \ldots + 3^n$
2. $5 + 10 + 20 + \ldots$ to n terms.
3. $1 + \frac{1}{3} + \frac{1}{9} + \ldots$ to 6 terms.

4. $x + x^2 + x^3 + \dots$ to 20 terms.

5. $1 + 4 + 16 + \dots$ to 5 terms.

6. $6 - 2 + \frac{2}{3} - \dots$ to 10 terms.

7. $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots + \frac{3}{10^6}$

8. $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \dots + 64$

9. A rubber ball is dropped from a height of 4.8 dm. It continuously rebounds, each time rebounding $\frac{3}{4}$ of the distance of the preceding fall. How much distance has it travelled when it strikes the ground for the sixth time?

10. What is the sum of the geometric series for which $a = 2, n = 5$ and $l = a_n = 32$?

11. Sum the series:

    (i) $.2 + .22 + .222 + \dots$ n terms.

    (ii) $3 + 33 + 333 + \dots$ n terms.

    (iii) $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3)$ $+ \dots$ n terms

    (iv) $1 + (1 + x)r + (1 + x + x^2)r^2 + (1 + x + x^2 + x^3)r^3$ $+ \dots$ n terms (x and r being proper fractions)

12. A business concern pays profit at the rate 15%, compounded annually. If an amount of Rs.200000 is invested with the concern now, what total amount

will become payable at the end of 5 years?

5.11. *Convergence and Divergence of Infinite Geometric Series*

Let us consider the following two infinite geometric series:-

(i) $2 + 6 + 18 + 54 + \ldots$

(ii) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

There is no last term in either of these two series. It is not possible to find a definite sum of series (i) because as we go on, the terms get bigger and bigger. Let us find the sum of the series (i) upto n terms:

$$S_n = \frac{2(3^n - 1)}{(3-1)} = \frac{2(1-3^n)}{-2} = -(1-3^n) = 3^n - 1$$

$\therefore$ the sum depends on 3^n.

Now $3^4 = 81, 3^{10} = 59049, 3^{20} = 3486784401$

Thus as n increases without bound, 3^n also increases without bound.

This situation is expressed by saying $\underset{n \to \infty}{Lt}\ 3^n = \infty$

This is read: "limit of 3^n as n tends to infinity is infinity."

(i) Divergent series

When the sum of an infinite series is infinity it is called a 'divergent series'.

It can be seen that $r^n \to \infty$ whenever $r > 1$

(In the above example $r = 3 > 1$).

In the general case, the sum of the infinite series:

$a + ar + ar^2 \ldots$, where $r > 1$, is given by:

$$S_\infty = \underset{n \to \infty}{Lt} \frac{a(r^n - 1)}{r-1}$$

$$= \underset{n \to \infty}{Lt} \left(\frac{a\,r^n}{r-1} - \frac{a}{r-1} \right)$$

$$= \frac{a}{r-1} \underset{n \to \infty}{Lt}\, r^n - \frac{a}{r-1}$$

$$= \infty - \frac{a}{r-1}$$

$$= \infty$$

(S_∞ is read as 'Sum to infinity' and it stands for $\underset{n \to \infty}{Lt}\ S_n$

i.e. the series is divergent in this case.

Take the series (ii) which is:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots, \text{ where } a = 1, r = \frac{1}{2}$$

$$r = \frac{1}{2} < 1$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{1 \cdot \left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$$

$$= 2\left(1 - \frac{1}{2^n}\right)$$

Clearly the sum depends upon the value of $\frac{1}{2^n}$

$$\text{Now } \frac{1}{2^5} = 0.03125, \frac{1}{2^{10}} \approx 0.000977$$

$$\frac{1}{2^{20}} \approx 0.000000954 \quad (\text{'}\approx\text{' means "nearly equal to"})$$

We notice that as n becomes greater and greater, $\frac{1}{2^n}$ becomes smaller and smaller. In other words we can say that $\frac{1}{2^n}$ can be made as small as we please by taking n sufficiently large. This state of affairs is expressed by saying that as n tends to infinity (∞) $\frac{1}{2^n}$ tends to 0.

Symbolically, $\underset{n \to \infty}{Lt} \frac{1}{2^n} \to 0$

which is read as "Limit of $\frac{1}{2^n}$, as n tends to infinity is zero".

For the series under consideration

$$S_\infty = \lim_{n \to \infty} S_n = \lim_{n \to \infty} 2\left(1 - \frac{1}{2^n}\right)$$

$$= 2\,(1 - 0) = 2$$

For the above series

$$r = \frac{1}{2} < 1$$

In fact, whenever, $|r| < 1$, $r^n \to 0$ as $n \to \infty$

e.g $\left(\frac{2}{3}\right)^n \to 0$, as $n \to \infty$

$$\left(\frac{-1}{5}\right)^n \to 0, \text{ as } n \to \infty$$

Using this fact we derive a general formula for S_∞ of an infinite geometric series in case $|r| < 1$

$$S_n = a + ar + ar^2 + \text{..... to } n \text{ terms,}$$

$$= \frac{a(1-r^n)}{1-r}$$

$$\therefore \quad \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1-r^n)}{1-r}$$

$$= \lim_{n \to \infty} a \left[\frac{1}{1-r} - \frac{r^n}{1-r}\right]$$

$$= \lim_{n \to \infty} \left(\frac{a}{1-r}\right) - \lim_{n \to \infty} a \left(\frac{r^n}{1-r}\right)$$

$$= \frac{a}{1-r} - 0 = \frac{a}{1-r}$$

$(\because r^n \to 0, \text{ as } n \to \infty)$

Thus $\quad S_\infty = \dfrac{a}{1-r}$

(ii) Convergent Series

When an infinite series has a finite definite sum, it is said to be a *convergent series.*

In case of a covergent series the sum S_∞ is given by the formula deduced above i.e.,

$$S_\infty = \frac{a}{1-r} \quad \text{when } |r| < 1$$

$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ is an example of convergent series.

Example 1

Show that the following series is a convergent series. Find its sum.

$$1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\ldots.$$

Solution

Given series is

$$1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\ldots.$$

$$a=1, \quad r=\frac{-3}{4}\div 1=\frac{-3}{4}$$

$$|r|=\left|\frac{-3}{4}\right|=\frac{3}{4}<1$$

$$r^n=\left(\frac{-3}{4}\right)^n=(-1)^n\left(\frac{3}{4}\right)^n$$

$$\lim_{n\to\infty} r^n=\pm 1\lim_{n\to\infty}\left(\frac{3}{4}\right)^n=\pm 1\,(0)=0$$

$$\therefore \quad S_\infty=\frac{a}{1-r}=\frac{1}{1-\left(-\frac{3}{4}\right)}=\frac{4}{7}$$

Hence the given series is a convergent series.

Example 2

Discuss the convergence or divergence of the series:-

(i) $2+4+8+16+32+\ldots$

(ii) $\frac{2}{3}+\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\dots$

Solution

(i) $2+4+8+16+32+\dots \quad a=2, r=4 + 2=2$

$$S_n=\frac{2(1-2^n)}{1-2}=\frac{2(1-2^n)}{-1}$$

$$=-2(1-2^n)=2^{n+1}-2$$

$$\lim_{n\to\infty} S_n=2^{n+1}-2=\infty$$

As $2^{n+1}\to\infty$ because $|r|=2>1$

Hence the series is a divergent series.

(ii) $\frac{2}{3}+\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\dots$

Here $a=2/3, r=\frac{1}{3} + \frac{2}{3}=\frac{1}{6} + \frac{1}{3}=\frac{1}{2}<1$

$$\therefore S_\infty=\frac{a}{1-r}=\frac{\frac{2}{3}}{1-\frac{1}{2}}$$

$$=\frac{2}{3}+\frac{1}{2}$$

$$=\frac{4}{3}$$

$\therefore$ the series is a convergent series.

Example 3

The sum of an infinite geometric series is 15 and the sum of the squares of its terms is 45. Find the series.

Solution

Let the series be:

$$a + ar + ar^2 + \dots$$

Then $S_\infty = \dfrac{a}{1-r}$

So $\dfrac{a}{1-r} = 15$ *or* $a = 15 - 15r$ ... (1)

The sum of the squares of the terms gives rise to the series:

$$a^2 + a^2r^2 + a^2r^4 + \dots$$

and its sum is given to be 45.

So $\dfrac{a^2}{1-r^2} = 45$

or $a^2 = 45\,(1 - r^2) = 45 - 45r^2$ ... (2)

Squaring (1) we get

$a^2 = (15 - 15r)^2$ ... (3)

From (2) and (3) we get

$$45 - 45\,r^2 = (15 - 15r)^2$$

$$= 225 - 450\,r + 225\,r^2$$

or $1 - r^2 = 5 - 10\,r + 5\,r^2$

or $0 = 5 - 10\,r + 5\,r^2 - 1 + r^2$

or $3\,r^2 - 5\,r + 2 = 0$

$$\therefore\ r = \frac{5 \pm \sqrt{25-24}}{6} = \frac{5 \pm 1}{6}$$

or $r = 1, \dfrac{2}{3}$

If $r = 1$, $a = 15 - 15.1 = 0$, so we cannot take $r = 1$

Hence we take $r = \frac{2}{3}$

$$a = 15 - 15 \,.\, \frac{2}{3} = 15 - 10 = 5.$$

Hence the series is $5 + \frac{10}{3} + \frac{20}{9} + \ldots$

5.12. Recurring Decimals

An interesting application of infinite geometric series is found in evaluation of recurring or periodic decimals.

Example 4

Express the recurring decimal fraction

$0.\overline{21}$ (or $0.\dot{2}\,\dot{1}$) as a common fraction.

(The bar ($\overline{}$) means that the numerals appearing under it are repeated endlessly)

Solution

$0.\overline{21}$ can be written as .21212121...

i.e. $0.\overline{21} = 0.21 + 0.0021 + 0.000021 + \ldots.$

This is a geometric series in which,

$$a = 0.21,\ r = \frac{0.0021}{0.21} = 0.01.$$

$\because |r| = 0.01 < 1$

$\therefore$ we can use the formula $\frac{a}{1-r}$ to find the sum of this infinite geometric series:

$$\therefore \quad 0.\overline{21} = 0.21 + 0.0021 + 0.000021 + \ldots$$

$$= \frac{0.21}{1-0.01} = \frac{0.21}{0.99} = \frac{21}{99} = \frac{7}{33}$$

Example 5

Evaluate $2.3\overline{124}$.

Solution

$$2.3\overline{124} = 2 + .3 + .0124 + .0000124 + \ldots$$

$$= 2.3 + [\, 0.0124 + 0.0000124 + \ldots]$$

$$= 2.3 + \frac{0.0124}{1-0.001}$$

$$= 2.3 + \frac{124}{9990}$$

$$= \frac{23}{10} + \frac{124}{9990}$$

$$= \frac{23 \times 999 + 124}{9990} \quad *$$

$$= \frac{23101}{9990}$$

---

* We may write

$$\frac{23 \times 999 + 124}{9990} = \frac{23\,(1000-1) + 124}{9990}$$

$$= \frac{23124 - 23}{9990}$$

This gives us the following rule for evaluating a recurring decimal in the form of a common fraction:

Rule: For the numerator of the required fraction write down the number formed by omitting the decimal point in the given decimal fraction and subtract from it the number formed by the non-recurring digits. For the denominator put down as many nines as there are digits in the recurring part of the given decimal, followed by as many zeros as there are non-recurring digits to the right of the decimal point.

Exercise 5.8

(a) Which of the following series are (i) divergent (ii) convergent?

1. $1 + 4 + 16 + 64 + \ldots$ 2. $6 + 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots$

3. $12 + 4 + \frac{4}{3} + \frac{4}{9} + \ldots$ 4. $18 - 6 + 2 - \frac{2}{3} + \ldots$

5. $6 + 12 + 24 + 48 + \ldots$ 6. $1 + 5 + 25 + 125 + \ldots$

(b) Find the sum of each of the following infinite geometric series.

7. $a = 3, r = 2/3$ 8. $a = 3, r = 3/4$

9. $a = 128, r = -\frac{1}{2}$ 10. $3 + 2 + \ldots$

11. $2 + 1 + \ldots$ 12. $\frac{1}{8} + \frac{1}{16} + \ldots$

(c) Find a common fraction for each of the following.

13. $0.\overline{81}$ 14. $0.\overline{31}$ 15. $2.\overline{410}$

16. $3.\overline{027}$ 17. $0.8\overline{33}$ 18. $0.5\overline{027}$

5.13. Harmonic Sequence

A sequence of numbers whose reciprocals form an arithmetic sequence, is called a harmonic sequence or harmonic progression (H.P.)

The following are a few examples of harmonic progressions.

(i) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

(ii) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

(iii) $\frac{1}{4}, \frac{1}{9}, \frac{1}{13}, \ldots$

(iv) $\frac{1}{5}, -1, -\frac{1}{7}, \ldots$

The standard form of an H.P. is:

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots$$

The general term or the nth term of this H.P. is

$$\frac{1}{a+(n-1)\,d}$$

Note: So far no formula has been obtained for finding the sum of *n* terms of the harmonic series

$$\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\ldots\ldots+\frac{1}{a+(n-1)\,d}$$

Example 1

Write three more terms of the harmonic progression given below:-

$$-\frac{1}{5}, -\frac{1}{3}, -1, \ldots$$

Solution

The corresponding A.P. is

$-5, -3, -1, \ldots$

Its next three terms are clearly

$1, 3, 5$

Therefore, three more terms of the H.P. are

$$1, \frac{1}{3}, \frac{1}{5}$$

Example 2

Find the 10th term of the given H.P

$$\frac{1}{8}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{7}, \ldots$$

Solution

$\frac{1}{8}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{7}, \ldots$ is an H.P.

Their reciprocals will be in A.P

$\therefore$ 8, 3, – 2, – 7, ... are in A.P

c.d. of the A.P. = 3 – 8 = – 5

$$\text{10th term of this A.P.} = 8 + (10 - 1)(-5)$$
$$= 8 + 9(-5)$$
$$= 8 - 45 = -37$$

Hence the 10th term of the given H.P. is – 1/37.

Example 3

The first term of an H.P. is – 1/3 and the 5th term is $\frac{1}{5}$. Find the 9th term.

Solution

First term = $-\frac{1}{3}$, 5th term = $\frac{1}{5}$

The first and the 5th terms of the corresponding A.P. will be – 3 and 5 respectively.

a_1 (of A.P) = – 3 a_5 (of A.P.) = 5

or $a = -3$ or $a + 4d = 5$

or $-3 + 4d = 5$

or $d = 2$

9th term of this A.P. = $a_9 = a + 8d$

$= -3 + 8 \times 2$

$= -3 + 16$

$= 13$

Hence the 9th term of the H.P. = $\frac{1}{13}$.

Exercise 5.9

1. Extend the harmonic sequence $1, \frac{2}{3}, \frac{1}{2}$ to five terms.

2. Find three more terms of the H.P. $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \dots$

3. Find five more terms of the H.P. $\frac{1}{3}, 1, -1, \dots$

4. Find the eighth term of the H.P. $3, 1, \frac{3}{5}, \dots$

5. Find the 10th term of the H.P. $\frac{1}{10}, \frac{1}{7}, \frac{1}{4}, 1, \dots$

6. Find 12th term of the H.P. $\frac{1}{3}, \frac{2}{9}, \frac{1}{6}, \dots$

7. The first term of a harmonic sequence is 4 and the

fourth term is $\frac{2}{5}$. Find the tenth term.

8. The second term of an H.P. is $\frac{1}{2}$ and the fifth term is $-\frac{1}{4}$. Find the 12th term.

9. In an H.P. $a_5 = \frac{1}{14}$, $a_8 = \frac{1}{2}$. Find a_{15}.

10. In an H.P. $a_7 = 1$ and $a_{10} = \frac{2}{5}$. Find a_{16}.

5.14. Harmonic Means

Definition

A number H is said to be the Harmonic Mean (H.M.) between two number a and b if a, H, b are in H.P.

$[a \neq 0, b \neq 0]$

In such a case $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in A.P.

$$\text{Therefore, } \frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{a+b}{2ab}$$

$$\text{Hence} \quad H = \frac{2ab}{a+b}$$

Def. $H_1, H_2, H_3, \ldots, H_n$ are said to be n Harmonic Means (H.M's) between a and b if

$a, H_1, H_2, H_3, H_4, \ldots, H_n, b$ are in H.P

i.e., $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots\ldots, \frac{1}{H_n}, \frac{1}{b}$ is an A.P.

$$\therefore \frac{1}{b} = \frac{1}{a} + (n+1)\,d,$$

$$\therefore \frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{ab\,(n+1)} \qquad \text{or} \qquad H_1 = \frac{ab\,(n+1)}{b\,(n+1) + (a-b)}$$

$$\frac{1}{H_2} = \frac{1}{a} + \frac{2\,(a-b)}{ab\,(n+1)} \qquad \text{or} \qquad H_2 = \frac{ab\,(n+1)}{b\,(n+1) + 2\,(a-b)}$$

... ...

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{ab\,(n+1)} \qquad \text{or} \qquad H_n = \frac{ab\,(n+1)}{b\,(n+1) + n\,(a-b)},$$

$$= \frac{ab\,(n+1)}{b + na}$$

Example 1

Find the H.M. of 5 and 7

Solution

Here $a = 5$, $b = 7$

$$\text{H.M.} = \frac{2ab}{a+b} = \frac{2 \times 5 \times 7}{5+7} = \frac{70}{12} = \frac{35}{6}$$

Example 2

Insert four H.M's between 4 and 20.

Solution

Let H_1, H_2, H_3 and H_4 be the four required harmonic means. Then

$4, H_1, H_2, H_3, H_4, 20$ are in H.P.

$\therefore \frac{1}{4}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{20}$ are in A.P.

$$\therefore \quad \frac{1}{20} = \frac{1}{4} + 5d \Rightarrow d = -\frac{1}{25}$$

$$\therefore \quad \frac{1}{H_1} = \frac{1}{4} - \frac{1}{25} = \frac{21}{100}$$

or $$H_1 = \frac{100}{21}$$

$$\frac{1}{H_2} = \frac{21}{100} - \frac{1}{25} = \frac{17}{100}$$

or $$H_2 = \frac{100}{17}$$

$$\frac{1}{H_3} = \frac{17}{100} - \frac{1}{25} = \frac{13}{100}$$

or $$H_3 = \frac{100}{13}$$

$$\frac{1}{H_4} = \frac{13}{100} - \frac{1}{25} = \frac{9}{100}$$

or $$H_4 = \frac{100}{9}$$

Hence $\frac{100}{21}, \frac{100}{17}, \frac{100}{13}, \frac{100}{9}$ are the four H.M's.

5.15. Relations between Arithmetic, Geometric and Harmonic Means.

(1) We know that for positive numbers a and b

$$A = \frac{a+b}{2}, G = \pm\sqrt{ab}, H = \frac{2ab}{a+b}$$

so that $$A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab$$

Therefore, $G^2 = (\pm\sqrt{ab})^2 = ab$

$\therefore \quad A \times H = G^2$

2. To show that A > G > H

We know that :

$$A = \frac{a+b}{2}, G = \pm\sqrt{ab}, H = \frac{2ab}{a+b}$$

A will be greater than *G* if

$$\frac{a+b}{2} > \pm\sqrt{ab}$$

or $\dfrac{(a+b)^2}{4} > ab$

or $(a+b)^2 - 4ab > 0$

or if $(a-b)^2 > 0$

which is always true so long as a and b are distinct positive numbers.

$\therefore A > G$

Again $G > H$ if

$$\pm\sqrt{ab} > \frac{2ab}{a+b}$$

or $\qquad ab > \dfrac{4a^2 b^2}{(a+b)^2}$

or $\qquad (a+b)^2 > 4ab$

or $\qquad (a-b)^2 > 0$

which is always true so long as a, b are distinct.

Hence $A > G > H$

Example

Find A, G and H for 9 and 25 and thus show that

$$A > G > H.$$

Solution

$$A = \frac{9+25}{2} = \frac{34}{2} = 17$$

$$G = \sqrt{9 \times 25} = 15$$

$$H = \frac{2 \times 9 \times 25}{9+25} = \frac{450}{34} = \frac{225}{17}$$

It is obvious that $A > G > H$

Exercise 5.10

1. Find the H.M. between $\frac{1}{8}$ and $\frac{1}{20}$
2. Find the H.M. between 9 and 11
3. Find the H.M. between $\frac{2}{5}$ and $\frac{3}{5}$
4. Find the H.M. between -6 and -8
5. Find two H.M's. between $\frac{5}{2}$ and $\frac{5}{11}$
6. Find four H.Ms between $\frac{7}{3}$ and $\frac{7}{11}$
7. Find A, G and H between 9 and 4 and show that $G^2 = A.H.$
8. Find A, G and H between -12 and -3 and verify that $AH = G^2$.

5.16 *Problems involving Sequences and Series*

Example 1

A club sold 1000 sealed tickets for a show. These tickets were numbered in succession from 1 to 1000. These tickets were selected at random by purchasers. Each purchaser paid the number of paisa equal to the number on the ticket. How much money did the club receive?

Solution

The number of paisa the club received is:-

$$S_{1000} = 1 + 2 + 3 + \ldots + 1000$$

$$a_1 = 1 \quad a_n = 1000, \ n = 1000$$

$$S_{1000} = \frac{n}{2}[a_1 + a_n] = \frac{1000}{2}[1 + 1000]$$

$$= 500\,(1000)$$

$$= 500500$$

$\therefore$ The club received = 500500 paisa = Rs.5005.

Example 2

The population of a certain town increases geometrically at the rate of 4% per year. If the present population is 200,000, what will be the population after 5 years?

Solution

$$a = 200{,}000, \ r = 4\% = .04, \ n = 6$$

$\because$ the term for the 5th year will be the 6th one)

Population after five years = $200{,}000\,(1 + .04)^5$

$$= 200{,}000 \times (1.04)^5$$

$$= 243320$$

Exercise 5.11

1. It is estimated that a certain property now valued at Rs.40,000 will depreciate as follows. Rs.1450 the first year, Rs.1400 the second year, Rs.1350 the third year and so on. Based on these estimates what will be the worth of the property 15 years hence ?

2. A rack in the form of an equilateral triangle is filled with balls so that there are 11 balls on a side. Another layer of balls is placed on this layer so that there are 10 balls on a side. A third layer, with 9 balls on a side, is placed on the second layer and so on. Thus a pyramid of balls is formed with 1 ball on the top. How many balls are there in the bottom layer? How many balls are there in the pyramid?

3. The sum of three numbers in A.P. is 36. Find the numbers if the sum of the squares of these numbers is 482.

4. A woman is offered a position at a starting salary of Rs.46,000 per year with annual increment of Rs.2500. How much would her total earnings amount to if she were to serve for 12 years ?

5. How far would a free falling body fall in 5 seconds if the number of meters travelled in each second is given by the sequence 5, 15, 25, 35, 45 ?

6. The population of a town has increased geometrically from 59,049 to 100,000. If this growth occurred in 5 years, what is the growth rate? [Hint: 9^5 = 59,049 : Use $a\,r^5 = 100{,}000$].

7. The number of bacteria in a culture increased geometrically from 64000 to 729000 in 6 days. Find the daily rate of increase if the rate is assumed to be constant.

8. If the population of a town increases geometrically at the rate of 5% per year and the present population is 300,000, what will be the population after 6 years from now ?

9. The number of bacteria in a culture increased geometrically from 320,000 to 2430,000 in 5 days. Find the daily rate of increase assuming the rate to be constant.

10. If three numbers in arithmetic progression are increased by 1, 4 and 15 respectively, the resulting numbers are in G.P. Find the original numbers if there sum is 6.

11. If three numbers in A.P. are increased by 9, 7 and 9 respectively, the resulting numbers are in G.P. Find

the original numbers if their sum is 3.

12. A Saving Bank pays profit at the rate of 5% compounded annually. What will be the total accumulation at the end of 5 years if Rs.2000 is deposited in the bank now ?

Chapter 6

Permutation, Combination and Introduction to Probability.

6.1 Factorial of a Positive Integer:

Let n be a positive integer, then the factorial of n is denoted by $n!$ and is defined by

$$n! = n(n-1)(n-2)(n-3) \ldots\ldots 3.2.1$$

e.g. $5! = 5.4.3.2.1 = 120$ {5! is read as factorial 5}

We define $0! = 1$

6.2 Permutation

Three pictures A,B,C are to be hung in a line on a wall. Following are the different orders in which they can be hung:

$$(A,B,C); (A,C,B); (B,A,C); (B,C,A); (C,A,B); (C,B,A)$$

Each of these arrangements is called a permutation of the three pictures.

Definition

An arrangement of a finite number of objects taken some or all at a time is called a permutation of these objects.

The number of ways of arranging 'n' objects taken 'r' at a time is denoted by ${}^{n}p_r$. In the following examples ${}^{n}p_r$ will be evaluated.

Example 1

How many numbers consisting of two digits can be formed from 2, 3, 5, 7? Each digit is to be used only once.

Solution

The following numbers of two digits can be formed from the given four digits 2, 3, 5, 7.

23 32 52 72
25 35 53 73
27 37 57 75

Explanation

From the above table we see that there are twelve possible formations. The tens place can be filled by any one of the four ; so there are four ways of filling the tens place. When the tens place has been filled up, there will be three digits left to occupy the units place. Any one of three digits can fill up the units place. So to each way of filling the tens place, there are three ways to fill the units place.

So total number of two-digits numbers = 4 × 3 = 12.

Example 2

How many arrangements can be made of four letters a, b, c, d taken three at a time?

Solution

As explained before, all such arrangements are:

abc	*acb*	*abd*	*adb*	*acd*	*adc*
bcd	*bdc*	*bac*	*bca*	*bad*	*bda*
cda	*cad*	*cab*	*cba*	*cbd*	*cdb*
dab	*dba*	*dbc*	*dcb*	*dac*	*dca*

As explained in example 1 above,

the number of arrangements = $4 \times 3 \times 2 = 24$

The examples explained above are illustration of the following principle:

"If, corresponding to one way of filling a place in m different ways there are n ways of filling another place, then the two places can be filled up in m × n different ways."

6.3 Formula For ^nP_r

^nP_r stands for number of permutations of n different objects taken 'r' at a time. This is equivalent to the number of ways in which r places can be filled up by n different objects.

In this section, we shall evolve the formula for ^nP_r. Suppose we have n objects and we want to determine the number of ways in which these objects can be arranged by taking r at a time. Any one of the n objects can fill up the first place, so there are n different ways of filling the first place. When the first place has been filled up in any one of the n ways, $(n - 1)$ objects remain for filling the second place. Therefore to each way of filling the first place, there are $(n-1)$ ways of filling the second place. Hence the first two places can be filled up in $n(n-1)$ ways. When the first two places are filled up in any one of the $n(n-1)$ ways, there will be $(n-2)$ objects left over. So for each way of filling the first two places, there will be $(n-2)$ ways of filling the 3rd place. Hence the first three places can be filled up in $n(n-1)(n-2)$ ways. Proceeding this way and noticing that the number of factors at any stage is the same as the number of places filled up, the number of ways in which 'r' places can be filled up is

$$n(n-1)(n-2) \text{ ...... } r \text{ factors}$$

The r th factor is $n-(r-1) = n-r+1$, so

$$^nP_r = n(n-1)(n-2) \text{ ...... } (n-r+1).$$

$$\text{Now } {}^nP_r = \frac{n(n-1)(n-2) \text{ .... } (n-r+1)(n-r)(n-r-1) \text{ .... } 3\cdot 2\cdot 1}{(n-r)(n-r-1) \text{ .... } 3\cdot 2\cdot 1}$$

which, in factorial notation, is

$$^nP_r = \frac{n\,!}{(n-r)\,!}$$

Corollary

For any positive integer n, $^nP_n = n\,!$

Proof

$$^nP_n = n\,(n-1)\,(n-2) \text{ ...... } n \text{ factors}$$

$$= n\,(n-1)\,(n-2) \text{ ...... } 1$$

$$^nP_n = n!$$

We define $^nP_o = 1$

Example 3

How many distinct four-digit numbers can be formed from the digits 1,2,3,4,5,6 if each digit is used only once?

Solution

The total number of digits is 6. Each number consists of four digits.

So the required number of four digit numbers is

$$^{6}P_{4}=\frac{6\,!}{(6-4)\,!}=\frac{6\,!}{2\,!}=360$$

Example 4

How many signals can be given with four different flags when any number of them may be hoisted at a time?

Solution

The number of flags = 4

Number of signals using 1 flag $= {}^{4}P_{1}=4$

Number of signals using 2 flags $= {}^{4}P_{2}=4.3=12$

Number of signals using 3 flags $= {}^{4}P_{3}=4.3.2=24$

Number of signals using 4 flags $= {}^{4}P_{4}=4.3.2.1=24$

Therefore the total number of signals

$$=4+12+24+24=64$$

Example 5

In how many ways can a set of four different mathematics books and five different physics books be placed on a shelf with a space for nine books, if all books on the same subject are to be kept together?

Solution

The four mathematics books can be arranged

among themselves in 4! = 24 ways.

The five physics books can be arranged among themselves in 5! = 120 ways.

To every one way of arranging the mathematics books, there are 5! ways of arranging physics book. The books in the two subjects can be arranged subject-wise in 2! = 2 ways. So the number of ways of arranging all the books

$= 4! \times 5! \times 2!$

$= (24)(120)(2)$

$= 5760$

Exercise 6.1

1. Find the values of:

   (i) ${}^{20}P_3$ (ii) ${}^{16}P_4$

   (iii) ${}^{12}P_5$ (iv) ${}^{10}P_7$ (v) ${}^{9}P_8$

2. Find the value of n when

   (i) ${}^{n}P_2 = 30$ (ii) ${}^{11}P_n = 990$

   (iii) ${}^{n}P_4 : {}^{n-1}P_3 = 9 : 1$

3. Prove from the first principle that

   (i) ${}^{n}p_r = n\,{}^{n-1}p_{r-1}$ (ii) ${}^{n}p_r = {}^{n-1}p_r + r\,{}^{n-1}p_{r-1}$

4. Find the number of signals that can be given by six

flags of different colours, using three flags at a time.

5. How many signals can be given by six flags of different colours when any number of them are used at a time?

6. How many words can be formed from the letters of the word "ARTICLE" using all the letters?

7. How many numbers can be formed from the digits 1, 3, 5, 7, 9 taken all at a time without repeating any digit?

8. How many numbers greater that 23000 can be formed from thedigits 1, 2, 3, 4, 5 when no digit is to be used more than once?

9. How many five digits numbers can be formed from the digits 2, 3, 5, 7, 9 if no digit is repeated but
   (i) the digits 2 and 9 are to be next to each other;
   (ii) the digits 2 and 9 are not to be next to each other?

10. How many six-digit numbers can be formed with the digits 0, 1, 3, 5, 7, 9? How many of these will have 0 in the tens place? No digit is to be repeated.

11. Eight books including two on mathematics are to be placed in a row on a table. In how many ways can the books be arranged so that two mathematics books are never together?

12. In how many ways can seven books be arranged on a shelf so that two particular books are always kept

together?

13. In how many ways can five books on English and three on Urdu be placed on a shelf so that the books on the same subject are always together?

14. From three books of Urdu, four of Islamiyat and six of Pakistan studies, how many sets of books can be chosen so that each set has two books in different subjects?

6.4 Permutation of Objects not all different

Suppose there are n objects of which there are 'p' *like* objects of one kind, 'q' like objects of the second kind and 'r' like objects of the third kind. If all the objects had been unlike; then the

p like objects could be arranged in $p!$ ways,

q like objects could be arranged in $q!$ ways

and r like objects could be arranged in $r!$ ways. So the total number of arrangements of p like objects, q like objects and r like objects simultaneously would be $(p!)(q!)(r!)$ ways. If X be the required number of permutation of the n objects in the present situation then we have

$$X.(p!)(q!)(r!)=n!$$

$$\text{So that } X=\frac{n!}{p!\cdot q!\cdot r!} \quad \ldots \quad (1)$$

The formula (1) can be generalised to cases with categories of like objects more than three. Formula (1) can be written in this way

$$X = \binom{n}{p, q, r}$$

Example 6

How many words can be formed out of the letters of the word "MISSISSIPPI" when all the letters are used in each word?

Solution

The word "MISSISSIPPI" contains 11 letters of which

S is repeated 4 times,

I is repeated 4 times,

P is repeated 2 times

and M comes only once.

therefore the total number of words that can be formed from the word MISSISSIPPI

$$= \binom{11}{4, 4, 2, 1}$$

$$= \frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!}$$

$$= 34650$$

Example 7

(i) How many numbers of six digits can be formed by the digits 2, 2, 3, 3, 3, 4 ?

(ii) How many of them will be lying between 400,000 and 430,000?

Solution

(i) There are six digits 2, 2, 3, 3, 3, 4 out of which

'2' is repated twice,

'3' is repeated thrice and

'4' comes only once.

Hence the numbers of six digits in this case are

$$= \binom{6}{2, 3, 1}$$

$$= \frac{6!}{2! \cdot 3! \cdot 1!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 3 \cdot 2} = 60$$

(ii) The numbers lying between 400,000 and 430,000 will have 42 in the extreme left position.

Thus there are left four digits 2,3,3,3, which can be arranged in the remaining four places.

Hence the required number of numbers

$$= \binom{4}{3, 1}$$

$$= \frac{4!}{3! \cdot 1!} = 4$$

Exercise 6.2

1. How many words can be formed out of the letters of the following words taken all together?

(i) PAKPATTAN (ii) PAKISTAN

(iii) MATHEMATICS

(iv) ASSASSINATION (v) INDEPENDENCE

2. How many numbers greater than 1,000,000 can be formed from the digits 2, 3, 3, 3, 0, 4, 2 ?

3. How many different numbers greater than three million can be formed with the digits 1, 1, 1, 1, 2, 2, 3 ?

4. In how many ways can thirteen cards be arranged if six of them are white, three are black and the remaining are red?

5. In how many of possible permutations of the letters of the word ADDING are the two D's:

   (a) together (b) separate?

6.5 Combination

In this chapter so far we have been discussing permutation, in which the order of a set of objects was important. But we often come across situations in which the order of the objects is immaterial.

If for instance there are 16 members of a touring Pakistan Cricket team out of which 11 are to play, they will chiefly be interested in random selection of 11 players rather than their order of selection. When a selection of objects is made without paying regard to the order of selection, it is called a *combination*. Thus *ABC, ACB, BAC, BCA, CAB and CBA* are different permutations, but they represent the same combination of three objects *A, B* and *C*.

The number of combination of 'n' things taken 'r' at a time is denoted as ${}^nC_r$ or $\binom{n}{r}$

6.6 Find The Number Of Combination Of n Dissimilar Objects Taken r At A Time i.e; Find ${}^nC_r$.

Let the required number of combination be ${}^nC_r$. Each of these combinations contains r objects, which can be permutated among themselves in $r!$ ways. So ${}^nC_r$ combinations will give rise to ${}^nC_r \cdot r!$ permutations. But the number of permutations of n objects taken r at a time is ${}^nP_r$

$$\therefore\ {}^nC_r \,.\, r\,! = {}^nP_r = \frac{n!}{(n-r)\,!}$$

$$\Rightarrow\ {}^nC_r = \frac{n!}{r!\,(n-r)\,!}$$

Cor.

$$\binom{n}{n} = {}^nC_n = \frac{n!}{n!\,(n-n)\,!} = \frac{n!}{n!\,0!} = 1$$

6.7 Complementary Combinations

If from n different objects a combination of r objects is made, then $n-r$ objects remain and they themselves form a combination of $n-r$ objects. Such a combination is called a complementary combination.

Example

Prove that ${}^nC_r = {}^nC_{n-r}$

First Proof

$$^nC_r = \frac{n!}{r!\,(n-r)\,!}$$

$$\text{and } ^nC_{n-r} = \frac{n!}{(n-r)\,!\,(n-n+r)\,!} = \frac{n!}{(n-r)\,!\,r!}$$

$$\therefore\ ^nC_r = {}^nC_{n-r}$$

Second Proof

^nC_r = Number of combinations of n objects taken r at a time.

Now by selecting r objects out of n, we are actually rejecting $(n-r)$ objects out of n.

Hence the number of selections of r objects out of n

$= {}^nC_r$

= Number of rejection of $(n-r)$ objects out of n.

$^nC_{n-r}$

$\therefore\ ^nC_r = {}^nC_{n-r}$

Example 8

If $^nC_8 = {}^nC_{12}$, find n.

Solution

We know that $^nC_r = {}^nC_{n-r}$

so $^nC_8 = {}^nC_{n-8}$ ... (i)

But $^nC_8 = {}^nC_{12}$ ... (ii)

Therefore from (i) and (ii) we have

$^nC_{n-8} = {}^nC_{12}$

$\therefore\ n - 8 = 12$

$\Rightarrow n = 20$

Example 9

How many diagonals can be drawn in a rectilinear figure of seven sides ?

Solution

There are in all seven vertices. Joining two vertices at a time,

$$\text{the number of line segments} = {}^7C_2 = \frac{7!}{2!\,5!} = 21$$

But these line segments include 7 sides.

Hence the number of diagonals = 21 – 7 = 14

Example 10

Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Solution

$$\text{L.H.S.} = {}^nC_r + {}^nC_{r-1}$$

$$= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{r\,(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)\,(n-r)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!\,(n+1)}{r\,(r-1)!\,(n-r+1)\,(n-r)!}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!}$$

$$= {}^{n+1}C_r = \text{R.H.S.}$$

Hence ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Exercise 6.3

1. Find the value of n, when

(i) ${}^nC_4 = {}^nC_5$ (ii) ${}^nC_2 = 66$

(iii) ${}^nC_6 = {}^nC_{12}$ (iv) $6\,{}^nC_4 = {}^nP_3$

2. Find the value of n and r when

(i) ${}^nC_r = 220$ and ${}^nP_r = 1320$

(ii) ${}^nC_r = 35$ and ${}^nP_r = 210$

(iii) ${}^{n+1}C_{r+1} : {}^nC_r : {}^{n-1}C_{n-1} = 11 : 6 : 3$

3. Find the number of diagonals of a ten sided figure.

4. Find the number of ways in which three persons can be selected out of eight.

5. In how many ways can a hockey eleven be selected out of fifteen players? How many of them will include a particular player?

6. Show that ${}^{15}C_{10} + {}^{15}C_{11} = {}^{16}C_{11}$

7. A committee of seven persons is to be chosen from eight men and ten women. How many of these will have

(i) Four women,

(ii) At the most four women,

(iii) At least four women.

8. Prove that ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$

6.8 Probability

Imagine that you have bought five tickets for a raffle and that 500 tickets all together have been sold. Assuming that any one of the 500 tickets is as likely as any other to be drawn for first prize, you would say that you had 5 chances in 500 or a chance of 1 in 100, of winning the first prize.

In this section we develop methods to deal with problems concerned with chance events.

Probability gives us a measure for the likelihood that something will happen. However it must be appreciated that probability can never predict the number of times that an occurance actually happens. But being able to quantify the likely occurance of an event is important because most of the decisions that affect our daily lives are based on likelihoods and not on absolute certainties. For example, if it is known that it is likely to rain on two days out of seven days at Murree, it does not mean that it will rain on four days in two weeks but you are likely to decide to take a raincoat with you if you are going to Murree for holidays.

6.9 Sample Space

A set consisting of all possible outcomes that may result from a random experiment is defined to be a sample space for the experiment and is denoted by the letter S. Each possible outcome is a member of the sample space.

For instance, the experiment of tossing a coin results in either of the two possible outcomes; a head (H) or a tail (T); landing on its edge or rolling away is not considered. The sample space for this experiment may be expressed in set notation as $S = \{H, T\}$. The sample space for tossing two coins once or tossing a coin twice, will contain four possible outcomes denoted by

$$S = \{HH, HT, TH, TT\}$$

Clearly S is the cartesian product of $A \times A$ where

The sample space for throwing a die is denoted by

$$S = \{1, 2, 3, 4, 5, 6\}$$

6.10 Event

An event is an individual outcome or any number of outcomes of a random experiment i.e., any subset of a sample space S of the experiment is called an event and it is denoted by capital letter, e.g. $A, B, C, \ldots$ etc.

Now consider the following situation:

A bag contains three white balls and two black balls, and one ball is removed from the bag. The sample space is the set $S = \{W_1, W_2, W_3, B_1, B_2\}$

If the event denoted by A is "the removal of a white ball". then $A = \{W_1, W_2, W_3\}$

We note that A is subset of S.

In general if E is an event, then $\{E\}$ is a subset of sample space $\{S\}$.

6.11 Mutually Exclusive Or Disjoint

Two events A and B are said to be mutually exclusive or disjoint if and only if they cannot both occur at the same time. That is they have no event in common. For instance when a coin is tossed, we get either a head (H) or a tail (T), but not both. Therefore the two events head and tail are mutually exclusive. When a dice is thrown, the possible outcomes 1,2,3,4,5,6 are mutually exclusive.

6.12 Equally likely

Two events A and B are said to be equally likely, when one event is as likely to occur as the other. In other words, each event should occur in equal number. For instance, when a fair coin is tossed, the head is as likely to appear as the tail.

6.13 Basic Definition Of Probability

The probability of an event, A, occurring is denoted by $P(A)$.

$$\text{Hence } P(A) = \frac{\text{Number of ways in which A occurs}}{\text{Number of ways in which all equally likely events occur}}$$

$$\text{or } P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of members of A}}{\text{Number of members of sample space S}} \quad \ldots (1)$$

where all members are equally likely. Here $n(A)$ denotes the number of elements of A. This is the basic definition of probablity.

As A is a subset of the sample space S, the numerator of the fraction (1) is always less than, or equal to, the denominator, so for any event A

$$0 \leq P(A) \leq 1$$

(i) If $P(A) = 1 \Rightarrow$ the event is an absolute certainty.

(ii) If $P(A) = 0 \Rightarrow$ the event is an absolute impossibility,

(iii) If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B).$$

For example, if one ball is taken from a bag containing only red balls, then

P (ball is red) = 1 and P (ball is blue) = 0

Example 11

A dice is thrown. What is the probability to get

(i) an odd number

(ii) an even number?

Solution

The sample space for this experiment is

$$S = \{1, 2, 3, 4, 5, 6\}$$

(i) Event (Occurance of an odd number)

$A = \{1, 3, 5\}$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

(ii) Event (Occurence of an even number)

$B = \{2, 4, 6\}$

$$\therefore P(B) = \frac{n\,(B)}{n\,(S)} = \frac{3}{6} = \frac{1}{2}$$

Example 12

A slip is picked out of ten slips numbered 1, 2, 3, 4, ..., 10. What is the probability to get the

(i) slip number which is divisible by 3

(ii) slip number which is not divisible by 3?

Solution

The sample space for this experiment is

$S = \{1, 2, 3, 4, ..., 10\}$

(i) Event (Occurence of a slip number divisible by 3)

$A = \{3, 6, 9\}$

$$\therefore P(A) = \frac{n\,(A)}{n\,(S)} = \frac{3}{10}$$

(ii) ***Event (Occurence of a slip number not divisible by 3)***

$B = \{1, 2, 4, 5, 7, 8, 10\}$

$$\therefore P(B) = \frac{n\,(B)}{n\,(S)} = \frac{7}{10}$$

Note: As $\frac{7}{10} = 1 - \frac{3}{10}$

$\therefore \; P(B) = 1 - P(A)$

The Probability that an Event does not happen

If, in a sample space of n equally likely occurences, the event A occurs r times, then there are $n - r$ occasions when A does not occur . The event A does not occur is denoted by

$\overline{A}$ (and read as "not A")

$$\text{Thus } P(A) = \frac{r}{n} \text{ and } P(\overline{A}) = \frac{n-r}{n} = 1 - \frac{r}{n}$$

$$\Rightarrow P(\overline{A}) = 1 - P(A)$$

Example 13

A fair coin is tossed three times. What is the probability that

(i) at the most one head

(ii) at least one head

appears?

Solution

The sample space S is represented by the following outcomes:

HHH, HHT, HTH, THH, HTT

THT, TTH, TTT.

$\therefore$ S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

As coin is fair, therefore each of these outcomes is equally likely.

(i) Let A represents the event that at the most one head appears

then A = {HTT, THT, TTH}

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

(ii) Let B represents the event that at least one head appears.

then $B = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}$$

Exercise 6.4

(1) For each of the following experiments, first list outcomes of sample space and of an event, and then find the probability.

(a) ***Experiment:*** A dice is rolled.

Events happening

(i) It shows an even number

(ii) It shows an odd number

(iii) Does not show an even number

(iv) It shows an even or an odd number.

(b) ***Experiment:*** Two coins are tossed simultaneously.

Events happening

(i) One head in two tosses

(ii) At least one head in two tosses.

(c) ***Experiment*** Draw randomly one slip from a box containing ten slips numbered 1,2,3,4 ..... 10.

Events happening

(i) Slip of seven number (ii) Slip of an odd number

(iii) Slip does not have even number

(iv) Slip of number less than the number 5.

(v) Slip of number bigger than 10

(vi) Slip of even or odd number.

(d) ***Experiment*** From the same box as in (c) above, randomly draw one slip and then another after replacing the first drawn slip.

Events happening

(i) First slip of number 1

(ii) Second slip of number 1

(iii) Both slips of number 1 (iv) first of even number and second of number 2.

(2) There are 40 students in a class of whom 25 are boys and 15 are girls.Theteacher writes the name of each student on a slip of paper. What is the probability that the name on the slip is (a) of a boy (b) of a girl?

(3) What is the probability of selecting a prime number from a list of natural numbers from 1 — 33 both inclusive, if every number has equal chance of being selected?

6.14 Estimating Probability

Example

The table below shows the result of rolling a die 60 times.

Event happening	**Tally marks**	**Frequency**
1	𝍸 IIII	9
2	𝍸 𝍸	10
3	𝍸 𝍸 II	12

4	𝍸 |||	8
5	𝍸 𝍸	10
6	𝍸 𝍸 |	11

From the table we have

$$\frac{\text{Number of times, an odd number, turning up}}{\text{Number of rolls}} = \frac{9+12+10}{60}$$

$$= \frac{31}{60} = \frac{1}{2} \text{ (Approximately)}$$

This ratio can be taken as an estimate of the probability because it shows that the chance of an odd number turning up is approximately 1 in 2.

You will notice that the greater the number of trials, the more accurate is the estimate of the probability.

Example

Below is a record of the number of rainy days in the month of August for the past 10 years.

22, 23, 24, 25, 24, 26, 25, 24, 23, 24

We can use the ratio below to estimate the probability of rain falling on a particular day in the month of August.

$$\frac{\text{Number of rainy days in the month of August for 10 years.}}{\text{Total number of days in the month of August for 10 years}}$$

Thus the probability =

$$\frac{22+23+24+25+24+26+25+24+23+24}{31 \times 10}$$

$$= \frac{240}{310}$$

$= 0.8$ (Approximately)

With this probability in mind, we can estimate the number of rainy days in a certain given season say, 20 days from 11th to 30th August as follows

Let x be the number of rainy days

$$\text{Then } \frac{x}{20} = 0.8$$

$$\Rightarrow x = 0.8 \times 20 = 16$$

so there would be 16 rainy days in the given period.

Exercise 6.5

1. One coin was tossed. The table below shows the result of 30 trials.

Event happening	***Tally marks***	***Frequency***
Head	𝍸 𝍸 ||||	14
Tail	𝍸 𝍸 𝍸 |	16

Study the table and use it to answer the following:

(a) How many times did 'heads' appear?

(b) How many times did 'tails' appear?

(c) How many times was the coin tossed?

(d) Estimate the probability of getting a head.

(e) Estimate the probability of getting a tail.

2. The probability that an egg will be broken during delivery from a farm to a supermarket is $\frac{1}{200}$. How

many broken eggs would there be in (a) 800 eggs (b) 1000 egges and (c) 5000 eggs?

3. In a light bulb factory, it is found that the probability of a bulb being defective is 0.12. Estimate how many defective bulbs there would be in (a) 1000 bulbs (b) 5000 bulbs (c) 10000 bulbs (d) 100000 bulbs.

4. In a traffic survey, the number of persons in each passing car was noted and the results tabulated as follows.

Number of persons in a car	1	2	3	4	5
Number of cars	50	40	10	20	5

   (a) Calculate, to one decimal place, the average number of persons per car.

   (b) Find probability that a car contained more than one person.

6.15 More About probability

In this section we shall have more examples on probability. We will find that set language can be useful. Let us take the definition of probability as an example.

Probability of an event happening

$$= \frac{\text{Number of favourable outcomes}}{\text{Number of equally possible outcomes.}}$$

Let us denote the set of all outcomes, favourable to an event E by E, the set of all equally possible out-comes by S and the probability of an event happening by P (E)

$$\text{then, } P(E) = \frac{n(E)}{n(S)}$$

Example

A die is rolled what is the probability of getting

(a) an even number

(b) a number greater than 4

(c) a number which is even and greater than 4

(d) a number which is even or greater than 4?

Solution

Let us solve this problem by using the following Venn diagram where E = {even numbers}

F = {numbers greater than 4}

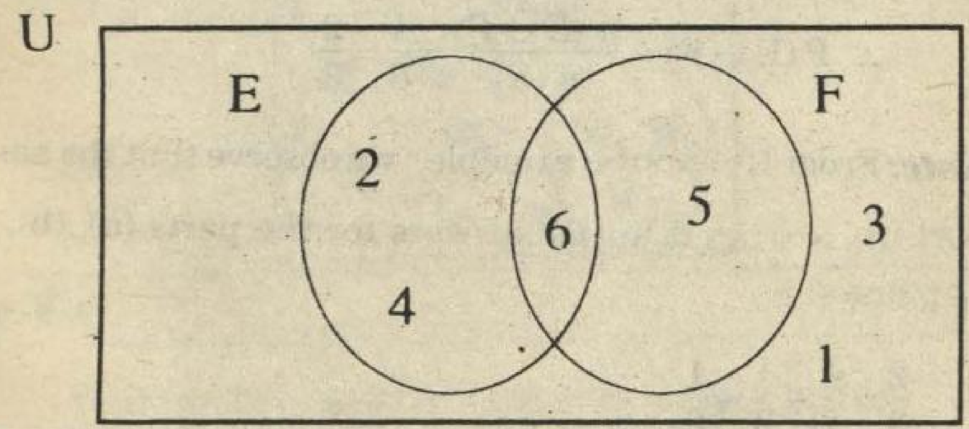

Fig. 6.1

The set of all equally possible outcomes is S = {1, 2, 3, 4, 5, 6}

(a) Events of getting an even number.

The set of favourable outcomes is

$E = \{2, 4, 6\}$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

(b) Event of getting a number greater then 4.

The set of favourable outcomes is $F = \{5, 6\}$.

$$\therefore P(F) = \frac{n(F)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

(c) Event of getting a number which is even and greater than 4.

The set of favourable outcomes is $E \cap F = \{6\}$

$$\therefore P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{1}{6}$$

(d) Event of getting a number which is even or greater than 4. The set of favourable outcomes is $E \cup F = \{2, 4, 5, 6\}$

$$\therefore P(E \cup F) = \frac{n(E \cup F)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

Note: From the above example . we observe that the answer for part (d) is related to the answers for the parts (a), (b), and (c) as follows:

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$$

$$\text{i.e } P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

You can establish this rule as follows.

From Venn diagram of above example , we have

$$\text{i.e } n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

$$\Rightarrow \frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

$$\Rightarrow P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

This rule is known as Addition Rule of Probability.

Example

A ball is drawn from a bag containing 2 black balls, 3 white balls and 5 green balls. What is the probability of getting a green ball or a black ball?

Solution

From Venn diagram in the set of all equally possible outcomes

$n(S) = 10$

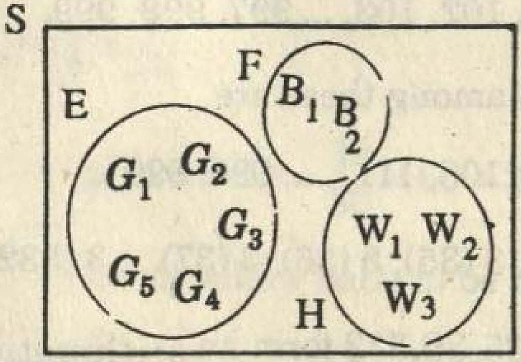

Fig. 6.2

∴ Probability of getting a green or a **black** ball is

$$P(E \cup F) = \frac{n(E \cup F)}{n(S)} = \frac{7}{10}$$

when $E = \{G_1, G_2, G_3, G_4, G_5\}$, $F = \{B_1, B_2\}$

Note: If E and F are disjoint sets, the rule reads

$$P(E \cup F) = P(E) + P(F)$$

Before proceeding further, let us summarize the rules about probability discussed so far.

(1) $P(E) = \dfrac{n(E)}{n(S)}$

(2) $P(\text{not } E) = 1 - P(E)$

(3) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

(4) $P(E \cup F) = P(E) + P(F)$, if E and F are disjoint.

Example

Find the probability that a 3 digit number chosen at random is divisible by 3.

Solution

The following are all the 3 digit numbers

100, 101, 102, 103, ... 997, 998, 999.

Multiples of 3 among these are

102, 105, 108, 111, ... 996, 999

or 3 (34), 3 (35), 3 (36), 3 (37) ... 3 (332), 3 (333).

The numbers 34, 35, 36, 333 form an arithmetic sequence with $a = 34$, $d = 1$, l = last term = 333. Hence total number of terms that is the number of integers divisible by 3 is given by

$$333 = 34 + \overline{n-1} \cdot 1 \text{ i.e. } n = 300.$$

Now $n(S) = 999 - 100 + 1 = 900$

and $n(E) = 333 - 34 + 1 = 300$

$$\therefore\ P(E) = \frac{n(E)}{n(S)} = \frac{300}{900} = \frac{1}{3}$$

Example

A bag contains 60 balls, some of which are red, some blue and the rest yellow. The probabilities of drawing a red ball and a blue ball are $\frac{1}{6}$ and $\frac{1}{5}$ respectively. Find

(a) The number of yellow balls

(b) The number of yellow balls to be removed so that the probability of drawing a yellow ball from the remaining balls in the bag is $\frac{9}{20}$.

Solution

(a) Probability of drawing a yellow ball

$$= 1 - \frac{1}{6} - \frac{1}{5} = \frac{19}{30}$$

The number of yellow balls $= \frac{19}{30} \times 60 = 38$

(b) Let the number of yellow balls to be removed $= x$

We have $\frac{38 - x}{60 - x} = \frac{9}{20}$

$\Rightarrow 38\,(20) - 20\,x = 9\,(60) - 9x$

$\Rightarrow 760 - 540 = 11x$

$\Rightarrow x = 20$

Thus the number of yellow balls to be removed = 20

Example

A man buys crates of eggs at Rs. 150 per crate. He sells each crate for Rs. 222,but there is probability of $\frac{1}{12}$ that a crate of eggs will be unfit for sale and he will lose Rs. 150. Find his average profit per crate.

Solution

The probability that a crate is fit for sale

$$= 1 - \frac{1}{12} = \frac{11}{12}.$$

Expected profit per crate

$$= \frac{11}{12}(222 - 150) = \frac{11}{12}(72) = \text{Rs } 66$$

Expected loss per crate

$$\frac{1}{12}(150) = \frac{25}{2} = \text{Rs. } 12.50$$

Average profit per crate = 66 – 12.50 = Rs. 53.50.

Exercise 6.6

1. The probability that your friend was born in January, June or July is $\frac{1}{4}$. What is the probability that he was not born in a month which begins with the letter J?
2. The probability that a certain team wins a foot-ball match is 0.7, the probability of a tie is 0.1. What is the probability of the team losing?
3. PIA is playing a football match against Railways. If

the probability that PIA will win is $\frac{4}{11}$ and the probability that Railways will win is $\frac{2}{11}$, find the probability that the match will be a draw.

4. A bag contains a number of balls, 30 of which are red and the remaining are blue. If a ball is chosen at random, the probability that it is red is 0.6. Find the number of blue balls in the bag.

5. We chose one number from among the first counting numbers 1,2,3, ... 20. What is the probability that the chosen number is divisible by 4 or by 5?

Hint: Let E be the event that the number is divisible by 4 and F be the event that the number is divisible by 5. Calculate $P(E \cup F)$.

6. The set $A = \{x : x \in I \text{ and } 1 \le x \le 200\}$
   (a) Find how many elements of A contain the digit 1 at least twice.
   (b) If an element of A is selected at random, write down the probability that it contains the digit 1 either once or not at all.

7. A market stall holder buys basket of vegetables at Rs 2 per kg. He sells each kg for Rs5, but there is a probability of $\frac{1}{18}$ that a basket is unfit for sale and he

will lose Rs 2 per kg. Find his average profit.

6.16 A Counting Principle

We have learnt that the probability of the happening of an event is $P(E) = \frac{n(E)}{n(S)}$

To find $n(E)$ and $n(S)$, we count the number of elements in each of the sets E and S. When this number is small, we can easily count by listing all the elements, but when the number of elements is very large, it is no longer practicable to list all the elements. We need another method. In this section, we present a fundamental method that is used very often in many counting and probability problems.

Example

Suppose we can go from city A to city B in three different ways (by car, by train, by aeroplane) and from city B to our home H in two different ways (by car, by tonga), then in how many different ways can we go from city A to our home H via city B?

Solution

We can count the different ways with the help of a diagram known as a tree diagram.

Starting from a point that represents city A, we draw 3 lines, one for each way by which we can go from A to B. From each of these lines, we can branch out in two other ways. So we draw two lines. The total number of ways of going from A to H via B is found to be 3 × 2 = 6. This is the total number of

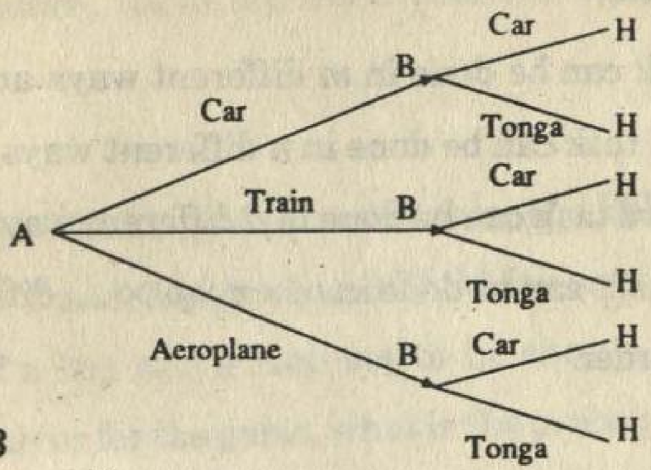
A
Car
Train
Aeroplane
B
B
B
Car
Tonga
Car
Tonga
Car
Tonga
H
H
H
H
H
H

Fig. 6.3

'branches' in the tree diagram.

Now suppose you can go from your home H to your school S in 3 different ways. (walk, cycle, bus). Observe how the tree diagram is continued. There are 3 × 2 × 3 = 18 ways of going from A to S via B and then to H. The tree diagram will be as follows.

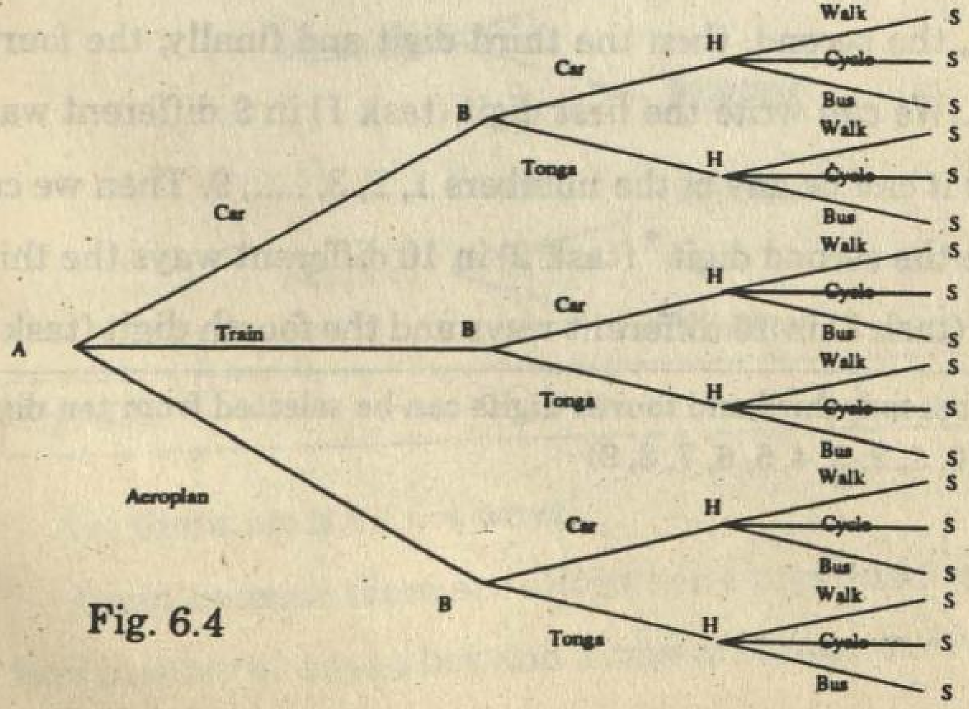
A
Car
Train
Aeroplan
B
B
B
Car
Tonga
Car
Tonga
Car
Tonga
H
H
H
H
H
H
Walk
Cycle
Bus
S
S
S

Fig. 6.4

This example illustrate the following fundamental principle of counting:

If one task can be done in m different ways and following this, a second task can be done in n different ways and following this, a third task can be done in p different ways and so on, then all the tasks can be done in $m \times n \times p \times \ldots$ different ways in the given order.

Example

In a certain town every telephone number consists of four digits, the first of which cannot be zero. How many different telephone numbers are possible?

Solution

We think of writing a telephone number by writing the first digit, the second, then the third digit and finally, the fourth digit. We can write the first digit (task 1) in 9 different ways since it can be any of the numbers 1, 2, 3, ..., 9. Then we can write the second digit * (task 2) in 10 different ways,the third digit (task 3) in 10 different ways and the fourth digit (task 4) in 10 different ways. By the fundamental principle of counting, we can do all four tasks in 9 × 10 × 10 × 10 = 9000 different ways. Therefore, there are 9000 possible 4 digit telephone numbers.

---

* second, third and fourth digits can be selected from ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

Example

Four boys Mushtaq, Maqsood, Rahim and Raheel and three men named Farid, Maqbool, and Mukhtar are to play a game of chess. If a boy and a man are to be chosen at random as representatives for the game, what is the probability that their names begin with the letter M?

Solution

The possible ways of choosing a pair of representatives whose names begin with the letter 'M' are as follows:

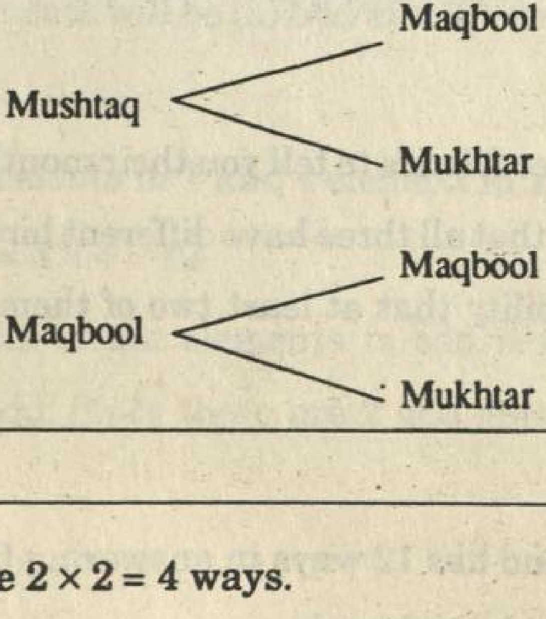

Fig. 6.5

i.e; there are 2 × 2 = 4 ways.

Again because there are altogether 4 boys and 3 men, the total number of ways a boy and a man are selected = 4 × 3 = 12 ways.

$$\text{We have } P(E) = \frac{n(E)}{n(S)} = \frac{2 \times 2}{4 \times 3} = \frac{4}{12} = \frac{1}{3}$$

Notice that we can write $P(E) = \frac{2}{4} \times \frac{2}{3} = \frac{1}{3}$ and interpret the result as the product of the probabilities of selecting one of the 2 boys whose name, begin with the letter 'M' from a group of 4 boys (i.e. $\frac{2}{4}$) and that of selecting one of the 2 men whose names begin with the letter 'M' from a group of 3 men, (i.e. $\frac{2}{3}$). We observe the following important result:

Suppose two experiments are to be carried out. If the probability of the event E_1 of 1st experiment is $P(E_1)$ and that of the event E_2 of 2nd experiment is $P(E_2)$, then the probability that both the events will happen is $P(E_1) \times P(E_2)$

Example

You ask three friends to tell you their months of birth. Find the probability that all three have different birth months. Find also the probability that at least two of them have the same birth month.

Solution

The 1st friend has 12 ways in answering the question and so do the 2nd and 3rd friends.

Hence $n(S) = 12 \times 12 \times 12 = 1728$

In the case that the three friends have different birth month, the 1st friend has 12 ways, the 2nd has 11 ways and the 3rd has 10 ways of answering the question.

Therefore $n\,(E) = 12 \times 11 \times 10 = 1320$

Hence the probability that three friends have different months is

$$P\,(E) = \frac{n\,(E)}{n\,(S)} = \frac{1320}{1728} = \frac{55}{72}$$

and the probability that at least two of them have the same birth month is

$$P\,(\text{not } E) = 1 - P\,(E) = 1 - \frac{55}{72} = \frac{17}{72}$$

Example

Let $T = \{3, 4, 5\}$ and $T' = \{4, 5, 6, 7\}$. If one element is chosen from each set at random, find the probability that the product of two chosen elements will be (a) odd and (b) even.

Solution

There are 3 elements in T and 4 element in T'

$$\therefore\ n\,(S) = 3 \times 4 = 12$$

(a) The product of the elements is odd if the elements chosen are both odd. Since there are 2 odd numbers in each set, we have

$$n\,(E) = 2 \times 2 = 4$$

$$\therefore\ P\,(E) = \frac{n\,(E)}{n\,(S)} = \frac{4}{12} = \frac{1}{3}$$

Alternative method

Probability of getting an odd number from Set T = $\frac{2}{3}$

Probability of getting an odd number from Set T' = $\frac{1}{2}$

$$\therefore \text{ Required probability} = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$$

(b) The probability that the product is even is

$$P(\text{not } E) = 1 - P(E) = 1 - \frac{1}{3} = \frac{2}{3}$$

Example

Three book shelves are placed inside a room. There are 5 books on shelf A in which 2 are mathematics books; 7 books on shelf B in which 3 are mathematics books; 5 books on shelf C in which 1 is mathematics book. If a person enters the room and picks one book at random from one of the shelves, find the probability that the book picked will be a mathematics book, assuming that the 3 book shelves are equally likely to be approached.

Solutions

Since the 3 books shelves have equal chances to be approached, hence they have each a probability of $\frac{1}{3}$ to be approached.

The probabilities of approaching and picking a mathematics book from shelves A, B and C are

$$\text{Shelf A} = \left(\frac{1}{3}\right)\left(\frac{2}{5}\right) = \frac{2}{15}$$

$$\text{Shelf B} = \left(\frac{1}{3}\right)\left(\frac{3}{7}\right) = \frac{1}{7}$$

$$\text{Shelf C} = \left(\frac{1}{3}\right)\left(\frac{1}{5}\right) = \frac{1}{15}$$

The probability that the person picks a mathematics book from any shelf is

$$= \frac{2}{15} + \frac{1}{7} + \frac{1}{15}$$

$$= \frac{36}{105}$$

$$\therefore \text{ Required probability} = \frac{12}{35}$$

Exercise 6.7

1. A restaurant offers two choices of soup, two different fish dishes and 3 different meat dishes. How many different meals consisting of soup, fish and meat are there? Draw a tree diagram to illustrate your answer.
2. A school club with 15 members has to select a president, a secretary and a treasurer. In how many different ways can this selection be made?
3. How many 3-digit numbers can be formed using the digits 1, 2, 3, 4, 5 if

(a) no digit is repeated,

(b) digit can be repeated,

(c) the number must begin with an even digit,

(d) the number must be an even number?

4. A 3-digit number is selected at random. Find the probability that the number selected

(a) begins with an even digit,

(b) is an even number,

(c) begins and ends with an even number.

5. A coin is tossed five times.

(a) How many different outcomes are there for this experiment?

(b) Assuming all outcomes are equally likely, find the probability that you get exactly one head among the five tosses.

6. (a) Write down the probability that a three-digit number chosen at random is divisible by 4.

(b) Find how many different 3 - digit numbers can be formed using the digits 1,2 and 3 if repetitions are allowed.

7. Two white and six red balls are placed in a bag. A ball is drawn at random from the bag, its colour is noted and the ball is replaced. A second ball is then drawn from the bag. Calculate the probability that the balls are

(a) both white (b) both red (c) of different colours.

Chapter 7

Principle of Mathematical Induction and Binomial Theorem

7.1. Introduction

In mathematics we deal with results which hold for all positive integers 'n'. Consider the formulas

(1) $$1+2+3+....+n=\frac{n\,(n+1)}{2}$$

(2) $$1^2+2^2+3^2+....+n^2=\frac{n\,(n+1)\,(2n+1)}{6}$$

and for a G.P.

(3) $$S_n=\frac{a\,(1-r^n)}{1-r}$$

Mathematical Induction is a method of proof of propositions / statements involving natural numbers. This method is a powerful tool for testing the validity of propositions involving positive integral variables.

7.2. Principle of Mathematical Induction

Statement

If a proposition $P(n)$ satisfies the conditions:

$C-1$. $P(n)$ is true for $n = 1$

$C-2$. Truth of $P(n)$ for $n = k$ implies its truth for $n = k + 1$,

then $P(n)$ is true for all positive integral values of n.

The use of the principle of Mathematical Induction is explained by means of the following worked examples:

Example 1

Prove that for any positive integer n,

$$1 + 3 + 6 + 9 + \dots + \frac{n}{2}(n + 1) = \frac{n}{6}(n + 1)(n + 2) \qquad \dots (A)$$

The proof consists of two parts.

1. We must show that the statement is true for $n = 1$.

We have $1 = \frac{1}{2}(1 + 1) = 1$

The statement is true for $n = 1$.

Thus $C-1$ is satisfied.

2. We must show if the statement is true for $n = k$, then it must be true for $n = k + 1$.

We begin by assuming that,

$$1 + 3 + 6 + \dots + \frac{k}{2}(k + 1) = \frac{k}{6}(k + 1)(k + 2) \qquad \dots (B)$$

Using (B) we want to prove that (A) is true for $n = k + 1$.

To prove this for $n = k + 1$, add $\frac{k+1}{2}(k + 2)$ to both members of (B) to get

$$\left[1+3+6+\ldots+\frac{k}{2}(k+1)\right]+\frac{k+1}{2}(k+2)$$

$$=\frac{k}{6}(k+1)(k+2)+\frac{(k+1)}{2}(k+2)$$

$$=\frac{(k+1)(k+2)}{2}\left[\frac{k}{3}+1\right]$$

$$=\frac{(k+1)(k+2)(k+3)}{6}$$

$$=\frac{\overline{(k+1)}}{6}(\overline{k+1}+1)(\overline{k+1}+2)$$

This equation is of the same form as the given equation (A) with n replaced by $k+1$.

Thus the truth of the given equation for $n = k$ implies its truth for $n = k + 1$, so that C–2 is also satisfied.

Hence by the Principle of Mathematical Induction, (A) is true for all positive integral values of n.

Note: It must not be inferred that the mathematical induction applies only when $P(n)$ is stated by means of an equation. The Induction can be applied in cases where no equation is involved. The following example will illustrate the case.

Example 2

Prove that 'if n is any integer then $x+y$ is a factor of $x^{2n}-y^{2n}$'.

Solution

Let $n = 1$, then $x^{2n}-y^{2n}=x^2-y^2$

Clearly $x^2 - y^2$ has $x + y$ as a factor.

Thus assertion is true for $n = 1$.

Hence C –1 is satisfied.

Now suppose that the statement is true for $n = k$.

So $x + y$ is a factor of $x^{2k} - y^{2k}$.

As $x + y$ is a factor of $x^{2k} - y^{2k}$ then

$$x^{2k} - y^{2k} = (x + y)\,Q$$

Where Q is the quotient when $x^{2k} - y^{2k}$ is divided by $x + y$, and

$$Q = x^{2k-1} - x^{2k-2}\,y + x^{2k-3}\,y^2 + \dots - y^{2k-1}$$

When $n = k + 1$ then

$x^{2n} - y^{2n}$ becomes $x^{2(k+1)} - y^{2(k+1)}$

Now $\quad x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - y^{2k+2}$

Adding and subtracting $x^2 y^{2k}$ to the right member to get

$$\begin{aligned} x^{2(k+1)} - y^{2(k+1)} &= x^{2k+2} - x^2 y^{2k} + x^2 y^{2k} - y^{2k+2} \\ &= x^2 (x^{2k} - y^{2k}) + y^{2k} (x^2 - y^2) \\ &= x^2 (x + y)\,Q + y^{2k} (x^2 - y^2) \\ &= (x + y)\,[x^2 Q + y^{2k} (x - y)] \end{aligned}$$

This shows that $x + y$ is a factor of $x^{2k+2} - y^{2k+2}$. Hence all the conditions of principle of Mathematical Induction are satisfied. Therefore the assertion is true for all positive integral values of n.

7.3 General Form of the Principle of Mathematical Induction

Sometimes a proposition $P(n)$ does not hold for first few values of n and becomes true for all positive integral values of n after a certain stage.

Consider the statement $n^2 > n + 3$

Then for $n = 1$, we have $1 > 4$ (false)

for $n = 2$, we have $4 > 5$ (false)

for $n = 3$, we have $9 > 7$ (true)

for $n = 4$, we have $16 > 7$ (true)

We find that the proposition is true for all values of n which are greater than 2

For the application of the Principle of Mathematical Induction in such cases, we give the statement in general form as follows:

If $P(n)$ is any proposition such that

C–1. $P(n)$ is true for $n = m$ where m is any natural number.

C –2: Truth of $P(n)$ for $n = k \geq m$ implies its truth for $n = k + 1$, then

$P(n)$ is true for all integral values of n greater than or equal to m.

This is known as the general form of the Principle of Mathematical Induction.

Example 1

Prove that $2^n > 2(n+1)$ for all $n > 3$.

Solution

for $n = 3$, $2^3 > 2(3+1)$

$\Rightarrow 8 > 8$ (false)

for $n = 4$, $2^4 > 2(4+1)$

$\Rightarrow 16 > 10$

We find that proposition is true for $n = 4$.

Assume that $P(n)$ is true for $n = k > 3$,

so that $2^k > 2(k+1)$ for $k > 3$.

Multiply both members of the inequality by 2 to get

$2 \cdot 2^k > 2 \cdot 2(k+1)$

$\Rightarrow 2^{k+1} > 2(k+1) + 2(k+1)$

$\Rightarrow 2^{k+1} > 2(k+1)$ (ignoring $2(k+1)\ k > 0$)

This shows that the proposition is true for $n = k + 1$.

Thus $2^n > 2(n+1)$ is true for $n > 3$.

Example 2

Prove that $3^n > n^3$ for all $n \geq 4$

for $n = 4$

$3^4 > 4^3$ which is true.

Suppose that the assertion is true for $n = k \geq 4$

Therefore $3^k > k^3$ ... (i)

Multiply both sides of (i) by 3 to get

$3 \cdot 3^k > 3 \cdot k^3$

$$\Rightarrow \quad 3^{k+1} > k^3 + k^3 + k^3$$

$$> k^3 + kk^2 + kk^2$$

$$> k^3 + 3\,k^2 + 3\,k \cdot 2 \qquad \text{As } k \geq 4$$

$$> k^3 + 3\,k^2 + 3k + 3k$$

$$> k^3 + 3k^2 + 3k + 1 \text{ as } 3k > 1$$

$$> (k + 1)^3$$

Thus the inequality holds for $n = k + 1$.

Hence, by the general Principle of Mathematical Induction,

$3^n > n^3$ for $n \geq 4$.

Exercise 7.1

Prove by Mathematical Induction that the following formulas hold for all positive integral values of n.

1. $1 + 2 + 3 + ... + n = \dfrac{n\,(n+1)}{2}$
2. $1 + 3 + 5 + ... + (2n - 1) = n^2$
3. $4 + 8 + 12 + ... + 4n = 2n\,(n + 1)$
4. $a + (a + d) + (a + 2d) + ... + (a + \overline{n-1}\,d)$

   $= \dfrac{n}{2}\,[\,2a + (n - 1)\,d]$
5. $\dfrac{1}{5 \cdot 8} + \dfrac{1}{8 \cdot 11} + \dfrac{1}{11 \cdot 14} + \ldots + \dfrac{1}{(3n+2)\,(3n+5)} = \dfrac{n}{5\,(3n+5)}$
6. $1^2 + 2^2 + ... + n^2 = \dfrac{n\,(n+1)\,(2n+1)}{6}$
7. $1^3 + 2^3 + ... + n^3 = \dfrac{n^2\,(n+1)^2}{4}$

8. $2^1 + 2^2 + 2^3 + \dots + 2^n = 2(2^n - 1)$

9. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

10. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

11. $\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \dots + \binom{n+2}{3} = \binom{n+3}{4}$

12. $\binom{5}{5} + \binom{6}{5} + \binom{7}{5} + \dots + \binom{n+4}{5} = \binom{n+5}{6}$

13. Show by Mathematical Induction that each of the following is an integer for any positive integer n.

(i) $\frac{7^n - 5^n}{2}$ (ii) $\frac{5^{2n-1} + 3^{2n-1}}{8}$ (iii) $\frac{n(n+1)}{2}$

14. Prove that:

(i) $n^2 > n + 4$ for $n \geq 3$

(ii) $n! > 3^{n-1}$ for $n \geq 5$

(iii) $3^n > 3n + 10$ for $n > 2$

15. If n is any positive integer then $x^n - y^n$ is divisible by $x - y$.

16. Prove that $2^n > n$ for every natural number n.

17. If n is any positive integer then $x + y$ is a factor of $x^{2n-1} + y^{2n-1}$.

18. (i) Prove that '2' is a factor of $5^{2n} - 3^{2n}$ for positive integral values of n.

(ii) $\frac{5^n - 2^n}{3}$ is an integer. for +ve integer values of n

19. Show that '5' is a factor of $3^{2n-1} + 2^{2n-1}$ where n is any positive integer.

20. Prove that for any positive integer,

$$\frac{1}{3}+\frac{1}{9}+\dots+\frac{1}{3^n}=\frac{1}{2}\left(1-\frac{1}{3^n}\right)$$

7.4. Sums of the Squares and the Cubes of the First n Natural Numbers

The symbol Σ (read as sigma) denotes the summation notation. We write the expression.

$$1+2+3+\dots+n$$

as $\sum_{k=1}^{n} k$ and read it as summation from $k = 1$ to $k = n$.

Similarly the expression

$$1^2+2^2+3^2+\dots+n^2$$

can be denoted by $\sum_{k=1}^{n} k^2$

Usually we write Σn, Σn^2, Σn^3 for the sums of first n natural numbers, the sum of the squares of first n natural numbers and the sum of the cubes of first n natural numbers respectively. Here n ranges from 1 to n.

(i) We know that the sum of first n natural numbers is

$$\sum_{k=1}^{n} k = 1+2+3+\dots+n=\frac{n\,(n+1)}{2}$$

(ii) To find the value of $\sum_{k=1}^{n} k^2$

Consider the identity written as

$$x^3 - (x-1)^3 = 3x^2 - 3x + 1$$

Put $x = 1, 2, 3, \ldots n$ in turn to get

$$1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$$

$$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$$

$$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$(n-1)^3 - (n-2)^3 = 3\,(n-1)^2 - 3\,(n-1) + 1$$

$$n^3 - (n-1)^3 = 3\,(n)^2 - 3\,(n) + 1$$

Adding the above results to get

$$n^3 = 3\,(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3\,(1 + 2 + 3 + \ldots + n) + n(1)$$

$$= 3\,\Sigma n^2 - 3\,\Sigma n + n \cdot 1$$

Hence

$$3\,\Sigma n^2 = n^3 + 3\,\Sigma n - n$$

$$= n^3 + 3\,\frac{n\,(n+1)}{2} - n \quad \text{as } \Sigma n = \frac{n\,(n+1)}{2}$$

$$= \frac{2n^3 + 3n^2 + 3n - 2n}{2}$$

$$= \frac{2n^3 + 3n^2 + n}{2}$$

$$= \frac{n\,(n+1)\,(2n+1)}{2}$$

Therefore $\Sigma\, n^2 = \dfrac{n\,(n+1)\,(2n+1)}{6}$

Thus $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n\,(n+1)\,(2n+1)}{6}$

(iii) Sum of the cubes of the first n natural numbers

Let us denote the sum by $\Sigma\, n^3$

$$\Sigma\, n^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Consider the identity written as

$$x^4 - (x-1)^4 = 4x^3 - 6x^2 + 4x - 1$$

put $x = 1, 2, 3 \ldots n$ in turn to get

$$1^4 - 0^4 = 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1$$

$$2^4 - 1^4 = 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1$$

$$4^4 - 3^4 = 4 \cdot 4^3 - 6 \cdot 4^2 + 4 \cdot 4 - 1$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$(n-1)^4 - (n-2)^4 = 4\,(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$

$$n^4 - (n-1)^4 = 4n^3 - 6 \cdot n^2 + 4 \cdot n - 1$$

Adding the above results to get

$$n^4 = 4\,(1^3 + 2^3 + 3^3 + \ldots + n^3) - 6(\,1^2 + 2^2 + 3^2 + \ldots + n^2\,)$$
$$+ 4(\,1 + 2 + 3 + \ldots + n\,) - n \cdot 1$$

$$= 4\,\Sigma\, n^3 - 6\,\Sigma\, n^2 + 4\,\Sigma\, n - n$$

Therefore $\quad 4\,\Sigma\, n^3 = n^4 + 6\,\Sigma\, n^2 - 4\,\Sigma\, n + n$

Replacing $\Sigma\, n^2$ and $\Sigma\, n$ by their values

$$4\,\Sigma n^3 = n^4 + \frac{6n\,(n+1)\,(2n+1)}{6} - \frac{4n\,(n+1)}{2} + n$$

$$= (n^4 + n) + n\,(n+1)\,(2n+1) - 2n\,(n+1)$$

$$= n\,(n^3 + 1) + n\,(n+1)\,[\,2n + 1 - 2\,]$$

$$= n\,(n+1)\,(n^2 - n + 1) + n\,(n+1)\,(2n-1)$$

$$= n\,(n+1)\,[\,n^2 - n + 1 + 2n - 1\,]$$

$$= n\,(n+1)\,(n^2 + n)$$

$$= n^2\,(n+1)^2$$

Therefore $\quad \Sigma n^3 = \dfrac{n^2\,(n+1)^2}{4}$

$$= \left[\frac{n\,(n+1)}{2}\right]^2 = [\Sigma n]^2$$

$$\text{So} = \sum_{k=1}^{n} k^3 = \left[\frac{n\,(n+1)}{2}\right]^2 = \left[\sum_{k=1}^{n} k\right]^2$$

The result also shows that the sum of cubes of the first n natural numbers is equal to the square of their sum.

Example 1

Sum to n terms the series

$$1 \cdot 5 + 5 \cdot 9 + 9 \cdot 13 + \ldots\ldots$$

Solution

Here $\quad T_n = (1 + \overline{n-1}\,4)\,(5 + \overline{n-1}\,4)$

$$= (4n - 3)\,(4n + 1)$$

$$= (16n^2 - 8n - 3)$$

$$S_n = T_1 + T_2 + T_3 + \ldots + T_n = \Sigma T_n$$

$$\Sigma T_n = 16\,\Sigma n^2 - 8\,\Sigma n - 3n$$

$$=16\,\frac{n\,(n+1)\,(2n+1)}{6} - 8\,\frac{n\,(n+1)}{2} - 3n$$

$$=\frac{8}{3}\,\mathrm{n}\,(\mathrm{n}+1)\,(2\mathrm{n}+1) - 4n\,(n+1) - 3n$$

$$=n\left[\frac{16}{3}n^2 + \frac{24}{3}n + \frac{8}{3} - 4n - 4 - 3\right]$$

$$=n\left[\frac{16}{3}n^2 + \frac{12}{3}n - \frac{13}{3}\right] = \frac{n}{3}\left[16n^2 + 12n - 13\right]$$

Example 2

Find the sum of the n terms of the series

$$1 + (1+2) + (1+2+3) + \ldots.$$

Solution

The nth term of the series is

$$T_n = 1 + 2 + 3 + \ldots + n = \Sigma n$$

$$=\frac{n\,(n+1)}{2}$$

Therefore $\Sigma T_n = \frac{1}{2}\Sigma(n^2+n) = \frac{1}{2}\Sigma n^2 + \frac{1}{2}\Sigma n$

$$=\frac{1}{2}\,\frac{n\,(n+1)\,(2n+1)}{6} + \frac{1}{2}\,\frac{n(n+1)}{2}$$

$$=\frac{1}{4}n\,(n+1)\left[\frac{2n+1}{3} + 1\right]$$

$$=\frac{1}{4}n\,(n+1)\left[\frac{2n+1+3}{3}\right]$$

$$=\frac{1}{6}n\,(n+1)\,(n+2)$$

Example 3

Sum to n terms the series with nth term $5n^3 + 3$.

Solution

Here $T_n = 5n^3 + 3$

put $n = 1, 2, 3 \ldots. n$ to get

$T_1 = 5 \cdot 1^3 + 3$

$T_2 = 5 \cdot 2^3 + 3$

$T_3 = 5 \cdot 3^3 + 3$

....

....

$T_n = 5 \cdot n^n + 3.$

Adding all the results

$$S_n = 5\,(1^3 + 2^3 + 3^3 + \ldots + n^3) + 3n$$

$$= 5\,\Sigma\, n^3 + 3n$$

The result can also be written directly as

$$\Sigma\, T_n = 5\Sigma\, n^3 + 3n$$

$$= 5\left[\frac{n\,(n+1)}{2}\right]^2 + 3n$$

$$= 5\,\frac{n^2\,(n+1)^2}{4} + 3n$$

$$= \frac{n}{4}\left[5n\,(n+1)^2 + 12\right]$$

$$= \frac{n}{4}\left[5n^3 + 10n^2 + 5n + 12\right]$$

Example 4

Sum to n terms the series

$$1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$$

Solution

Here $T_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \Sigma n^2$

$$= \frac{n\,(n+1)\,(2n+1)}{6}$$

$$= \frac{1}{6}\left[2n^3 + 3n^2 + n\right]$$

Therefore $\Sigma\, T_n = \frac{1}{6}\left[2\Sigma n^3 + 3\Sigma n^2 + \Sigma n\right]$

$$= \frac{1}{6} \cdot 2\left[\frac{n\,(n+1)}{2}\right]^2 + \frac{1}{2}\,\frac{n\,(n+1)\,(2n+1)}{6} + \frac{n\,(n+1))}{12}$$

$$= \frac{1}{12}\,n(n+1)\,[n\,(n+1) + (2n+1) + 1]$$

$$= \frac{1}{12}\,n\,(n+1)^2\,(n+2)$$

Exercise 7.2

Sum the following series upto n terms:

1. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots.$
2. $4^2 + 5^2 + 6^2 + \ldots.$
3. $2^2 + 4^2 + 6^2 + \ldots.$
4. $1^2 + 3^2 + 5^2 + \ldots.$
5. $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \ldots.$
6. $1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \ldots.$

7. $1^3 + 3^3 + 5^3 + ...$

8. $1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + ...$

9. $2 \cdot 1^2 + 3 \cdot 2^2 + 4 \cdot 3^2 + ...$

10. $1^3 + 5^3 + 9^3 + ...$

11. $1^2 + 5^2 + 9^2 + ...$

12. $1 \cdot 3^2 + 2 \cdot 4^2 + 3 \cdot 5^2 + ...$

13. $3 \cdot 2^2 + 4 \cdot 3^2 + 5 \cdot 4^2 ....$

Given the nth term of the series, find the sum of $2n$ terms in each case.

14. $4n^2 + 5n + 1.$

15. $2n^3 + 3n + 1.$

16. $n^3 + 3n^2 + 2n.$

7.5. Binomial Theorem

Let a, b be any real numbers and n a natural number:

If n is any positive integer then the binomial theorem states that the expansion of $(a + b)^n$ is given by

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + \binom{n}{n} b^n \quad \text{..........(A)}$$

We shall prove this theorem by applying the Principle of Mathematical Induction:

The theorem is true for $n = 1$ as

$$(a + b)^1 = a + b$$

So C –1 is satisfied.

Suppose that the theorem is true for $n = k$, then by this supposition.

$$(a+b)^k = \binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \dots + \binom{k}{r}a^{k-r}b^r + \dots + \binom{k}{k}a^0 b^k \qquad \text{..........} \text{(B)}$$

We now prove that the theorem is true for $n = k + 1$.

Multiply both sides of (B) by $(a + b)$ to get

$$(a+b)^{k+1}$$

$$= a\left[\binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \dots + \binom{k}{r}a^{k-r}b^r + \dots + \binom{k}{k}b^k\right]$$

$$+ b\left[\binom{k}{0}a^k b^0 + \binom{k}{1}a^{k-1}b + \binom{k}{2}a^{k-2}b^2 + \dots + \binom{k}{r}a^{k-r}b^r + \dots + \binom{k}{k}b^k\right]$$

Collecting like terms on the R.H.S. we have

$$(a+b)^{k+1} = \binom{k}{0}a^{k+1} + \left[\binom{k}{0} + \binom{k}{1}\right]a^k b + \left[\binom{k}{1} + \binom{k}{2}\right]a^{k-1}b^2$$

$$+ \dots\dots + \left[\binom{k}{r-1} + \binom{k}{r}\right]a^{k+1-r}b^r + \dots + \binom{k}{k}b^{k+1}$$

We know that

$$\binom{k}{0} = \binom{k+1}{0} = 1$$

and $\binom{k}{k} = \binom{k+1}{k+1}$, $\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$

Therefore

$$(a+b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \binom{k+1}{2}a^{k-1}b^2$$

$$+ \dots + \binom{k+1}{r}a^{k+1-r}b^r + \dots + \binom{k+1}{k+1}b^{k+1}$$

This shows that the theorem is true for $n = k + 1$. Thus C –2 is also satisfied. Hence the theorem is valid for any positive integer.

7.6. Characteristics of the Binomial Expansion

1. In every term the sum of the exponents of a and b is n.
2. The first term is a^n and the second term is $n\,a^{n-1}b$. In each other term the exponent of a decreases progressively by 1 and the exponent of b increases progressively 1.
3. The coefficients of the terms equidistant from the ends of the expansion are the same.
4. The number of terms in the expansion is one more then the index n.
5. The general term in the expansion of $(a+b)^n$ is denoted by T_{r+1}. So

$$T_{r+1}=\binom{n}{r}a^{n-r}\cdot b^r=\frac{n\,!}{(n-r)\,!\,r\,!}a^{n-r}\cdot b^r$$

The exponent of a in T_{r+1} is $n-r$ and that of b is r. The numerator of the coefficient is $n\,!$ while the denominator is $(n-r)!\,r!$

After simplification we can write

$$T_{r+1}=\frac{n\,(n-1)\,(n-2)\,....(n-r+1)}{1\cdot 2\,....\,r}a^{n-r}\cdot b^r$$

Example 1

The term involving b^5 in the expansion of $(a+b)^8$ is

$$\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5!} a^3 b^5 \text{ or } 56\, a^3 b^5$$

Example 2

Give the complete Binomial expansion of $(2x - y)^5$.

Solution

By the binomial formula

$$(2x - y)^5 = \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4 (-y) + \binom{5}{2}(2x)^3 (-y)^2$$

$$+ \binom{5}{3}(2x)^2 (-y)^3 + \binom{5}{4}(2x)(-y)^4 + \binom{5}{5}(-y)^5$$

$$= 1 \cdot 2^5 \cdot x^5 - 5 \cdot 2^4 \cdot x^4 \cdot y + \frac{5 \cdot 4}{2 \cdot 1} \cdot 2^3 \cdot x^3 \cdot y^2 - \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} \cdot 2^2 \cdot x^2 \cdot y^3$$

$$+ \frac{5 \cdot 4 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 1} \cdot 2 \cdot x \cdot y^4 - \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot y^5$$

$$= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$$

Example 3

Find the 6th term in the expansion of $(x + 3y)^{10}$

Solution

Here $n = 10, r = 5,$

Therefore $$T_6 = \binom{10}{5}(x)^{10-5} \cdot (3y)^5 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} x^5 \cdot 3^5 \cdot y^5$$

$$= 61236\, x^5 y^5$$

Example 4

Find the coefficient of x^5 in the expansion of $\left(2x^2 - \dfrac{3}{x}\right)^{10}$

Solution

Let T_{r+1} involve x^5

Therefore $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Here $n = 10, a = 2x^2, b = -\dfrac{3}{x}$

$$T_{r+1} = \binom{10}{r}\left(2x^2\right)^{10-r} \cdot \left(-\frac{3}{x}\right)^r$$

$$= \binom{10}{r} 2^{10-r} \cdot x^{20-3r} \cdot (-3)^r$$

Exponent of x in $T_{r+1} = 20 - 3r$

For the desired result $20 - 3r = 5$

Therefore $r = 5$

Hence x^5 occurs in T_6

Thus $T_6 = \binom{10}{5} \cdot 2^5 \cdot x^5 \cdot (-3)^5$

$$= -\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \cdot 2^5 \cdot 3^5 \cdot x^5$$

$$= -1959552\, x^5$$

Therefore Coefficient of $x^5 = -1959552$

Example 5

Find the term independent of x in the expansion of $\left(2x^2 - \dfrac{1}{x}\right)^{12}$

Solution

Let T_{r+1} be the term independent of x.

Therefore $$T_{r+1} = \binom{12}{r}\left(2x^2\right)^{12-r} \cdot \left(-\frac{1}{x}\right)^r$$

$$= \binom{12}{r} 2^{12-r} \cdot x^{24-2r} \cdot (-1)^r \cdot x^{-r}$$

$$= \binom{12}{r} 2^{12-r} \cdot (-1)^r \cdot x^{24-3r}$$

Exponent of $x = 24 - 3r$

For the desired result $24 - 3r = 0$

So $r = 8$

Hence T_9 is independent of x

$$T_9 = \binom{12}{8} \cdot 2^4 \cdot (-1)^4 = \frac{12\,!}{8\,! \cdot 4!} \cdot 2^4$$

$$= 7920$$

Exercise 7.3

1. Complete the following statements:

(i) Polynomial consisting of two terms is called ......

(ii) The number of terms in the expansion of $(a+b)^{12}$ is ......

(iii) The index in $(a+b)^7$ is ......

(iv) The middle term in the expansion of $(a+b)^{20}$ is ......

(v) In the binomial expansion the coefficients from the beginning and end are ......

(vi) The numerical coefficient of the 6th term in the expansion of $(x+y)^{16}$ is......

2. Exapand the following:

(i) $(2x+y)^6$ (ii) $\left(x+\frac{1}{x}\right)^7$

(iii) $\left(2x+\frac{1}{x}\right)^8$ (iv) $\left(2x^2-\frac{1}{3x}\right)^6$

(v) $\left(\frac{x}{2}-\frac{2}{y}\right)^4$

Find the indicated term in the expansion of

3. $\left(3x+\frac{1}{2x}\right)^{10}$: middle term.

4. $(1-x)^{2n}$: n th term.

5. $\left(\sqrt{x}+\frac{2}{\sqrt{x}}\right)^{12}$: eighth term

6. $\left(2x-\frac{x^2}{4}\right)^7$: fifth term.

Find the coefficient of

7. x^{20} in the expansion of $\left(x^2+\frac{1}{2x}\right)^{16}$

8. x^5 in the expansion of $\left(2x^2-\frac{1}{3x}\right)^{10}$

Find the term independent of x in the expansion of

9. $\left(2x^2+\frac{1}{x}\right)^9$

10. $\left(\sqrt{x}+\frac{1}{3x^2}\right)^{10}$

Find the specified term in the expansion of

11. $(x-y)^9$: term involving x^5

12. $(x+\sqrt{y})^{12}$: term involving y^4

13. $\left(\sqrt{x}+y^2\right)^8$: term involving $x^{5/2}$

14. $\left(x^2-\dfrac{y^3}{2}\right)^{17}$: term involving y^{15}

15. $\left(x^3+\dfrac{1}{x}\right)^7$: term involving x^9

16. Expand $\left(1-x+x^5\right)^5$, in ascending powers of x.

17. Simplify $(a+b)^5+(a-b)^5$.

18. Expand $(1.04)^5$, upto four places of decimal by the binomial formula.

19. The amount due on rupees P at the end of n years at the rate of i (expressed in decimal) is $P\,(1+i)^n$. Find to the nearest two places of decimal the amount due, given that

(i) $P = 100, i = .03, n = 4$

(ii) $P = 400, i = .03, n = 12$

(iii) $P = 600, i = .04, n = 10$

20. Show that the sum of the coefficients in the expansion of $(1+x)^n$ is 2^n. Hence show that the sum of the cofficients in the expansion of $(1+x)^5$ is 32.

21. Find the approximate values of

(i) $(2\cdot03)^5$ (ii) $(1\cdot05)^6$

(iii) $(1 \cdot 02)^{10}$

22. Show that the sum of the odd coefficients is equal to the sum of the even coefficients in the binomial expansion and each of them is equal to 2^{n-1}

7.7. Binomial Theorem for Rational Index: Approximations

If n is a positive integer then Binomial formula is

$$(a+b)^n = a^n + n\,a^{n-1}b + \frac{n\,(n-1)}{2!}\,b^2 + \dots + b^n \qquad \dots \quad (1)$$

and there are $n + 1$ terms in R.H.S. of (1)

Now consider the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2!}\,x^2 + \frac{n\,(n-1)\,(n-2)}{3!}\,x^3 + \dots + x^n$$

If n is not a positive integer then the expression on the R.H.S. of (2) does not terminate and has infinite terms. Thus, when n is any rational number negative or a fraction then

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2!}\,x^2 + \dots \infty \qquad \dots \quad (2)$$

This is possible only if $|x| < 1$. The proof of this result is beyond the scope of this book.

The series on the R.H.S. of (2) is called related binomial series.

Binomial series can be used to find expressions approximately equal to the given expressions under given conditions.

Example 1

Find the first four terms of the binomial expansion

$$\left(9+\frac{4}{x}\right)^{\frac{1}{2}}$$

Solution

Let us assume the validity of the binomial expansion for $n = 1/2$, we write $9+\frac{4}{x}$ as, $9\left(1+\frac{4}{9x}\right)$ and apply the theoreum

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\dots$$

$$\left(9+\frac{4}{x}\right)^{1/2} = 9^{1/2}\left(1+\frac{4}{9x}\right)^{1/2}$$

$$=3\left[1+\left(\frac{1}{2}\right)\frac{4}{9x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\cdot 1}\left(\frac{4}{9x}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{4}{9x}\right)^3+\dots\right]$$

$$=3\left[1+\frac{2}{9x}-\frac{1}{8}\frac{16}{81x^2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2\cdot 1}\frac{64}{729x^3}+\dots\right]$$

$$=3\left[1+\frac{2}{9x}-\frac{2}{81x^2}+\frac{4}{729x^3}+\dots\right]$$

The first four terms will be an approximation of the complete expansion if $\left|\frac{4}{9x}\right| < 1$ or $|x| > \frac{4}{9}$

Example 2

Compute a 3 significant digit approximation for $(8.2)^{1/3}$.

Solution

$$(8.2)^{1/3} = (8+.2)^{1/3}$$

$$= 8^{1/3}\left(1+\frac{.2}{8}\right)^{1/3}$$

$$= 2\,(1+.025)^{1/3}$$

$$= 2\left[1+\frac{1}{3}(.025)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(.025)^2+\dots\right]$$

$$= 2\left[1+.00833-\frac{1}{9}(.000625)+\dots\right]$$

$$= 2\,[\,1.00833-.00007\,]$$

$$= 2\,[\,1.00826\,] = 2.01652$$

Example 3

Evaluate $\sqrt{40}$ by binomial formula.

Solution

$$\sqrt{40}=\sqrt{36+4}=6\left(1+\frac{4}{36}\right)^{1/2}$$

$$= 6\left[1+\frac{1}{2}\cdot\frac{1}{9}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{1}{9}\right)^2+\dots\right]$$

$$= 6\left[1+.05556-\frac{1}{8}\cdot\frac{1}{81}+\dots\right]$$

$$= 6\,[\,1+.05556-.00154]$$

$$= 6\,[\,1.05402\,] = 6.32412$$

Example 4

If x be so small that its square and higher powers be neglected then evaluate

(i) $\frac{\sqrt{4+x}}{4-x/3}$ (ii) $\frac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}}$

Solution

(i) $$\frac{\sqrt{4+x}}{4-x/3}=\frac{2\,(1+x/4)^{1/2}}{4\,(1-x/12)}$$

$$=\frac{1}{2}\left[1+\frac{x}{8}\right]\left[1-\frac{x}{12}\right]^{-1}$$

$$=\frac{1}{2}\left[1+\frac{x}{8}\right]\left[1+\frac{x}{12}\right]$$

$$=\frac{1}{2}\left[1+\frac{x}{8}+\frac{x}{12}\right]$$

$$=\frac{1}{2}\,[1+\frac{3+2}{24}x]=\frac{1}{2}\left[1+\frac{5}{24}x\right]$$

(ii) $$\frac{(1-x)^{1/2}(9-4x)^{1/2}}{(8+3x)^{1/3}}=\frac{(1-x)^{1/2}\,3\left(1-\frac{4}{9}x\right)^{1/2}}{2\left(1+\frac{3}{8}x\right)^{1/3}}$$

$$=\frac{3\left(1-\frac{1}{2}x\right)\left(1-\frac{2}{9}x\right)}{2\left(1+\frac{3}{8}x\right)^{1/3}}$$

$$=\frac{3}{2}\left[1-\frac{1}{2}x-\frac{2}{9}x\right]\left[1+\frac{3}{8}x\right]^{-1/3}$$

$$=\frac{3}{2}\left[1-\frac{9+4}{18}x\right]\left[1-\frac{1}{8}x\right]$$

$$=\frac{3}{2}\left[1-\frac{13}{18}x\right]\left[1-\frac{1}{8}x\right]$$

$$=\frac{3}{2}\left[1-\frac{1}{8}x-\frac{13}{18}x\right]$$

$$=\frac{3}{2}\left[1-\frac{9+52}{72}x\right]$$

$$=\frac{3}{2}\left[1-\frac{61}{72}x\right]$$

$$=\frac{3}{2}-\frac{61}{48}x$$

7.8. Evaluation by use of Binomial Series

Identify the series

$$1+\frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\quad\ldots \qquad \text{(A)}$$

and find its sum.

Let the series be identical with the expansion of $(1+x)^n$

$$(1+x)^n=1+nx+\frac{n\,(n-1)}{2!}x^2+\ldots\quad\ldots \qquad \text{(B)}$$

Comparing (A) and (B), we get

$$nx=\frac{3}{4}\quad\ldots \qquad (1)$$

$$\text{or}\quad n^2x^2=9/16\quad\ldots \qquad (2)$$

$$\frac{n\,(n-1)}{2}x^2=\frac{15}{32}\quad\ldots \qquad (3)$$

From (2) and (3) by division

$$\frac{n-1}{2n}=\frac{15}{32}\cdot\frac{16}{9}=\frac{5}{6}$$

Therefore $\frac{n-1}{n}=\frac{5}{3}$

Hence $3n-3=5n$

$n=-3/2$

put this value of n in (1) to get

$x=-1/2$

$$\text{Thus series}=\left(1-\frac{1}{2}\right)^{-3/2}=\left(\frac{1}{2}\right)^{-3/2}$$

$$=2^{3/2}=2\sqrt{2}$$

Exercise 7.4

Expand the following upto the fourth term

1. $(1-2x)^{-1}$
2. $(1+x)^{-2}$
3. $(1-2x)^{-2}$
4. $(4+x)^{1/2}$
5. $(5-x)^{1/3}$
6. $(1-x)^{-2}$
7. $(1-x)^{-3}$
8. $\frac{1}{\sqrt{1-x}}$
9. $\sqrt[3]{1+x}$

Evaluate the following to the nearest hundredth.

10. $\sqrt{23}$
11. $\sqrt[3]{29}$
12. $\sqrt[4]{65}$
13. $\sqrt{52}$
14. $\sqrt[5]{33}$
15. $\sqrt{12}$
16. $\sqrt{80}$

17. If x be so small that its square and higher powers be neglected then show that

(i) $\dfrac{1+2x}{(1-3x)^2} = 1 + 8x.$

(ii) $\dfrac{(1+x)^2}{(1+2x)^3} = 1 - 4x.$

(iii) $\dfrac{2+x}{(1-x)^3} = 2 + 7x.$

(iv) $\left(\dfrac{4-x}{1-x}\right)^3 = 16\,(4 + 9x)$

(v) $\dfrac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} = 4\left(1 - \dfrac{5}{6}x\right)$

18. Identify the following series and find the sum of each

(i) $1 - \dfrac{1}{2^2} + \dfrac{1 \cdot 3}{2!} \cdot \dfrac{1}{2^4} + \ldots$

(ii) $1 + \dfrac{1}{3} + \dfrac{1}{2!} \cdot \dfrac{1}{3} + \dfrac{1.5}{3!} \cdot \dfrac{1}{3^2} + \ldots$

(iii) $1 - \dfrac{3}{2.4} - \dfrac{1.3}{4^2 \cdot 2!} \cdot \dfrac{1}{4} - \dfrac{1.3.5}{4^3 \cdot 3!} \cdot \dfrac{1}{8} + \ldots$

19. If $y = \dfrac{3}{4} + \dfrac{3.5}{4.8} + \dfrac{3.5.7}{4.8.12} + \ldots\ldots \infty$

show that $y^2 + 2y - 7 = 0$

20. If $y = \dfrac{1}{2^2} + \dfrac{1.3}{2!} \cdot \dfrac{1}{2^4} + \dfrac{1.3.5}{3!} \cdot \dfrac{1}{2^6} + \ldots.,$

then $y^2 + 2y - 1 = 0$

21. If $2y = \frac{1}{2^2} + \frac{1.3}{2!} \cdot \frac{1}{2^4} + \frac{1.3.5}{3!} \cdot \frac{1}{2^6} + \dots \infty$,

    then $4y^2 + 4y - 1 = 0$

22. Find the approximate percent increase in the area of a circle if its radius is increased in length from 50 to 50.5 dms.

23. Find the approximate percent increase in the volume of a sphere if its radius is increased in length from 100 to 101 dm. (Volume of the sphere is $\frac{4}{3}\pi r^3$)

Chapter 8

Fundamentals of Trigonometry

8.1 Introduction

The word 'trigonometry' is derived from three Greek words - 'treis' (three), 'goni' (angle) and 'metron' (measurement). So literally this word means 'measurement of the trigon or triangle'. Technically it denotes that branch of mathematics which deals with solution of triangles. However, in recent past, scope of Trigonometry has been much widened and, apart from solving triangles, many other problems requiring a use of trigonometric functions are also discussed in the same subject.

Trigonometry is of two types, Plane Trigonometry and Spherical Trigonometry. In the former type we are concerned with angles, triangles and other figures which lie in a plane where as in the latter type, we deal with spherical triangles, that is, triangles which lie on a sphere and sides of which are, therefore, circular arcs. Since the earth is of spherical shape, spherical trigonometry is applied in many problems connected with the human abode — problems relating to latitudes and longitudes, distance between different cities, finding direction of 'Qibla' etc. It is also used in Astronomy which deals with positions and motions of heavenly bodies.

Muslim astronomers like Abu Rehan Al-Beruni, Ibnul-Hatham and even Alkhawarzimi have made valuable contributions to the development of both plane and spherical trigonometry, specially to the latter branch.

In this book we shall deal with Plane Trigonometry only.

8.2 Measurement of Angles

In elementary (Euclidian) geometry * an angle is defined as the union of two non-collinear rays which have a common end-point. In trigonometry also an angle is regarded as the union of two rays but here the mode of measuring angle is different from that of elementary geometry. The main points of difference are the following:-

(a) In elementary geometry an angle has a positive measure only whereas in trigonometry angles have positive as well as negative measures. This situation is similar to measurement of line-segments in elementary geometry and coordinate geometry. In the former type of geometry one deals with un-

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* Elementary geometry is called Euclidian Geometry because the first systematic treatise on the subject was written by the famous Greek Mathematician Euclid (330 B.C. to 275 B.C.)

directed line-segments and length of a line-segment is always a positive number but in the latter geometry one deals with directed line-segments whose lengths may be positive as well as negative. We may say that elementary geometry deals with undirected angles but trigonometry deals with directed angles.

(b) In trigonometry an angle may have any measure. It is not restricted to the range 0° to 180°.

8.2.1 The Degree Measure

This is the common way of measuring angles. If a circle is divided into 360° congruent (equal in length) parts, the angle subtended by one part at the centre of the circle is called a *degree.*

An angle of 90° is called a right angle. 60th part of a degree is called a *minute* and 60th part of a minute is called a *second.* (Remember that these minutes and seconds are quite different from those used in connection with time). We have thus

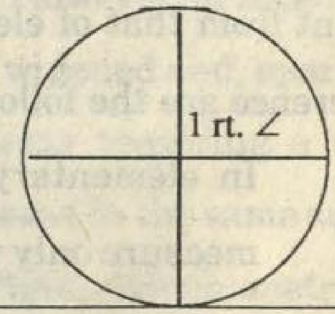

Fig. 8.1

1 rt. angle = 90° (read as 90 degrees)
1° = 60′ (read as 60 minutes)
1′ = 60″ (read as 60 seconds)

This system of measurement is called *sexagesimal system*[*]

8.2.2. The Radian Measure or Circular Measure

In Radian Measure the circle is involved. In higher mathematics the radian measure or circular measure is often employed. In this system, the unit of measurement of angles is the radian which is also defined with the help of a circle.

Definition

A *radian* is the measure of the central angle of an arc of a circle whose length is equal to the radius of the circle.

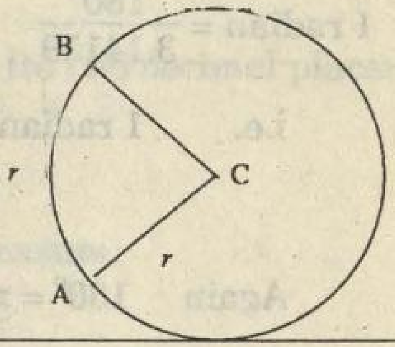

Fig. 8.2

In fig 8.2 $\overset{\frown}{AB}$ is an arc of a circle whose length is equal to r, the radius of the circle.

$\therefore\ m\ \angle\ ACB = 1$ radian .

8.2.3 Relation between Degree and Radian Measures

The circumference of a circle is $2\pi r$ units. Therefore, length of a half circle is πr units.

Since an arc of length r subtends at the centre an angle of 1 radian, therefore, an arc length πr will subtend an angle of $\frac{1}{r}\times\pi r$ i.e., π radians at the centre. Thus

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[*] 'Sexagesimal' is derived from the Latin word 'sexigesims' which means 60th part.

$$\pi \text{ radians} = 180^o \quad \ldots \quad (1)$$

(We have used here the fact that the measure of the central angle of a circular arc is proportional to its length.)

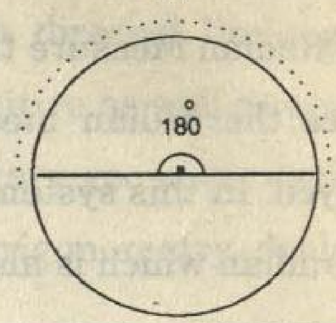


Fig. 8.3

Eq. 8.1 gives:

$$1 \text{ radian} = \frac{180^o}{3.14159}$$

$$\text{i.e.} \quad 1 \text{ radian} = 57^o.29578 \text{ (nearly)}$$

$$= 57^o\, 17'\, 45''$$

$$\text{Again} \because \quad 180^o = \pi \text{ radians}$$

$$= 3.14159 \text{ radians}$$

$$\therefore \quad 1^o = 0.01745 \text{ radian (nearly)}$$

Note

When the measure of an angle is expressed in terms of radians, the word 'radians' is generally omitted. Thus 'an angle of $\frac{\pi}{2}$' means, 'an angle of $\frac{\pi}{2}$ radians'.

By using relation (1) we can form the following table:-

Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
Degrees	30	45	60	90	120	135	150	180	270	360

Example 1

Express in degrees: (i) $\frac{5\pi}{3}$ radians (ii) 2.5793 radians.

Solution

$$\pi \text{ radians} = 180^\circ$$

(i) $$\therefore \frac{5\pi}{3} \text{ radians} = \frac{180}{\pi} \times \frac{5\pi}{3} = 300^\circ$$

(ii) $$1 \text{ radian} = 57^\circ.29578$$

$$\therefore\ 2.5793 \text{ radians} = 57.29578 \times 2.5793 \text{ (degrees)}$$

$$= 147^\circ.78301$$

$$= 147^\circ.78 \text{ (to two decimal places).}$$

Example 2

Express $36^0\ 47'\ 12''$ in radian measure.

Solution

$$36^0\ 47'\ 12'' = 36^\circ.78667$$

$$= 36.78667 \times .01745 \text{ (radians).}$$

$$= 0.64193 \text{ radian}$$

$$= 0.64 \text{ radian (to two decimal places)}$$

Important Note

$\because$ π radians = 180°

$\therefore$ 1 radian = $180^\circ/\pi$

Since π is a constant (the constant ratio of the circumference of a circle to the length of its diameter), therefore the radian is a constant unit and its value does not depend upon the radius of the circle with the help of which we define it.

8.2.4 Relation between the length of a Circular Arc and the Radian Measure of its Central Angle.

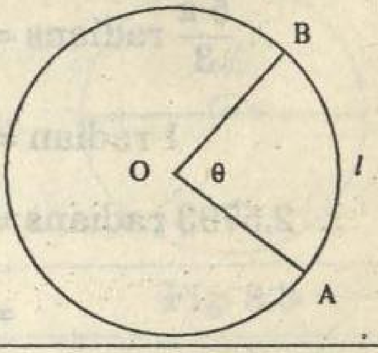


Fig. 8.4

Let l be the length of a circular arc AB, θ be the radian measure of its central angle and r the radius of the circle. Then the ratio of l to the circumference of the circle is the same as the ratio of θ to 2π ($\because 360^\circ = 2\pi$)

$$\therefore \quad \frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\text{i.e.,} \quad \theta = \frac{l}{r} \quad \ldots \qquad (1)$$

This gives the radian measure of the central angle of an arc of length l.

From (1) we also have

$$l = r\,\theta \quad \ldots \qquad (2)$$

Which is the required relation between the length of a circular arc and the radian measure of its central angle.

Note

When using formula (1) or (2), the angular measure should be expressed in radians if it is not already so expressed.

Example 3

What is the length of an arc of a circle of radius 5 c.m. whose central angle is of 140°?

Solution

$$140^\circ = 140 \times .01745 \text{ radians}$$

$$= 2.443 \text{ radians}$$

$$l = r\,\theta$$

$$= 5 \times 2.443$$

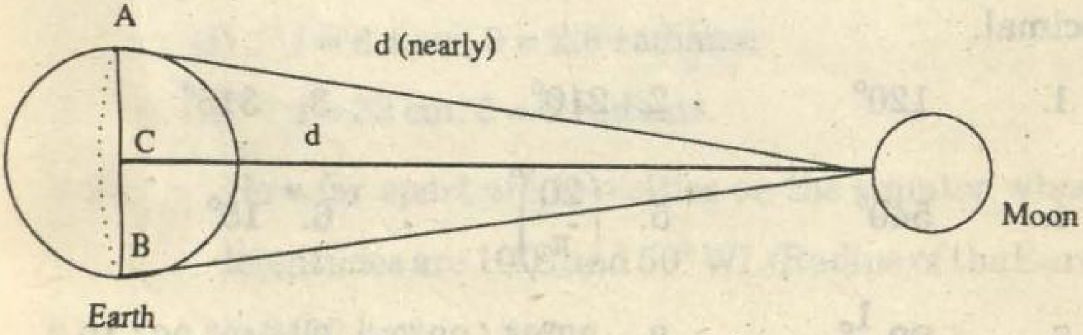
$$= 12.215$$

$$= 12.2 \text{ c.m. nearly}$$

Example 4

A spaceman lands on the Moon and observes that the Earth's diameter subtends an angle of $1^\circ\,54'$ at his place of landing. If the Earth's radius is 6400 km, find the distance between the Earth and the Moon.

Solution



Let O be in the place of landing on the Moon and $\overline{AB}$ the diameter of the Earth. Let d (km) be the distance between the Moon and the Earth. Imagine a circle of radius OA with centre at the Moon. Since the angle subtended by the Earth is very small, the circular arc intercepted by the Earth will be, in length, very nearly equal to the diameter of the Earth. Also $\overline{OA} \equiv d$. We have thus

$$r = d$$

$$\theta = 1^\circ 54' = 1.9 \times .01745 \text{ (radian)}$$

$$= .03315 \text{ (radian)}$$

$$l = 2 \times 6400 = 12800 \text{ km}$$

The relation $l = r\,\theta$ gives

$$r = d = \frac{l}{\theta} = \frac{12800}{.03315} = \frac{1280000000}{3315} = 386000 \text{ km (in round fig.)}$$

(More accurate figure for the distance is 384400 km)

Exercise 8.1

(a) Convert each of the following measures to radians. Use $\pi = 3.1416$ and the answer should be correct to two places of decimal.

1. 120°
2. 210°
3. 315°
4. 540°
5. $\left(\frac{20}{\pi}\right)^\circ$
6. 15°
7. $22\frac{1}{2}^\circ$
8. $37^\circ 25' 28''$
9. $42^\circ 36' 12''$

(b) Convert each of the following radian measures to degree measure. (Give answer correct upto two places of decimal):-

10. $\frac{5\pi}{4}$
11. $\frac{2\pi}{3}$
12. $3\,\pi$

13. 5.52 14. $\frac{1}{\pi}$ 15. $\frac{4}{3\pi}$

16. 4.52 17. $2\frac{3}{4}$ 18. $\frac{7}{4}\pi$

19. Through how many radians does the minutes hand of a clock turn in one hour? The hour hand in one hour?

20. What is the radian measure of the central angle of an arc 50m long on a circle of radius 25m?

21. Find the radian measure of the angle at the centre of a circle of radius 6 cm which cuts off an arc:-

(a) 12 cm long (b) 18.84 cm long

22. Find the length of the arc cut off on a circle of radius 3 cm by a central angle of

(a) 2 radians (b) 5.5 radians (c) $\frac{5\pi}{6}$ radians.

23. Find the radius of the circle when

(i) $l = 8.4$ cm, $\theta = 2.8$ radians;

(ii) $l = 33$ cm, $\theta = 6$ radians.

24. How far apart are two cities on the equator whose longitudes are 10°E and 50° W? (Radius of the Earth is 6400 km).

(***Hint:*** The angular distance between the cities is 60°).

25. Show that the area of a sector of a circular region of radius r is $\frac{1}{2}r^2\theta$, where θ is the radian measure of

the central angle of the sector.

26. Find the area of a sector with a central angle of 0.5 radian in a circular region whose radius is 2m.

[***Hint:*** Assume: Area of a circular region: area of a circular sector = 2 π : measure of the angle of the sector].

27. The area of a sector of a circular region is 200 sq cm and its central angle measures 2 radians. What is the radius of the circular region?

28. The Moon subtends an angle of 0.5° as observed from the Earth. Its distance from the earth is 384400 km. Find the length of the diameter of the Moon.

29. The Sun is about of 1.496×10^8 km from the Earth If the angle subtended by the Sun on the surface of the earth is 9.3×10^{-3} radians approximately, what is the diameter of the Sun?

8.3 The General Angle

As stated earlier the trigonometric way of associating a measure with an angle is different from that adopted in elementary geometry.

Consider a coordinate plane in which coordinate axes X′OX and Y′OY have been specified. Suppose a ray OP, initiallly coincident with the ray OX, revolves about the end point O in the counter clockwise direction. $\overrightarrow{OP}_1$, $\overrightarrow{OP}_2$, $\overrightarrow{OP}_3$, $\overrightarrow{OP}_4$ and $\overrightarrow{OP}_5$

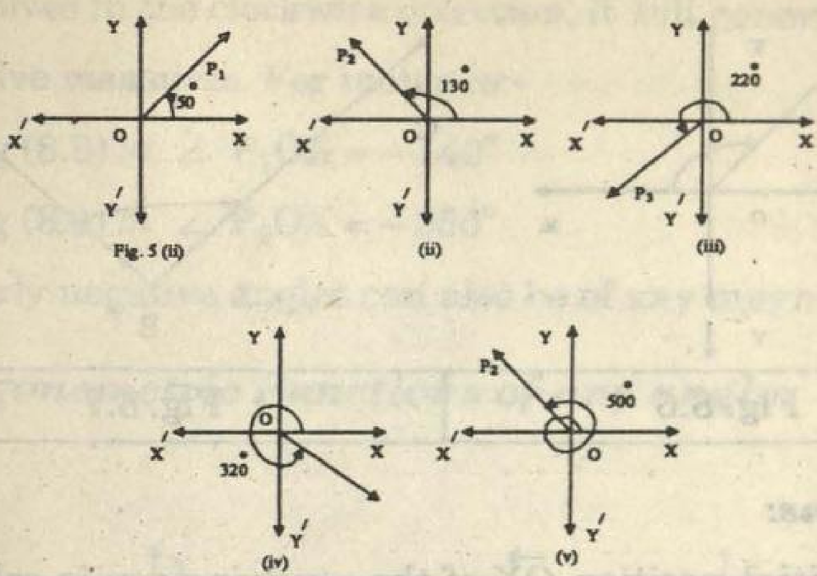


Fig. 8.5

are some of its subsequent positions (as in fig. 8.5). If the amounts of rotations from the initial position for the above mentioned positions are respectively

$50^\circ, 130^\circ, 220^\circ, 320^\circ, 500^\circ$,

We say that:-

$m\angle P_1OX = 50^\circ$, $m\angle P_2OX = 130^\circ$,

$m\angle P_3OX = 220^\circ$, $m\angle P_4OX = 320^\circ$,

$m\angle P_5OX = 500^\circ$.

It means that angles are measured positively from $\overrightarrow{OX}$ in the counter-clockwise direction.

The revolving ray may continue to revolve after completing one full revolution. It may thus generate angles of measures greater than 360° as we see in fig 8.5 (v) that $m\angle P_5OX = 500^\circ$.

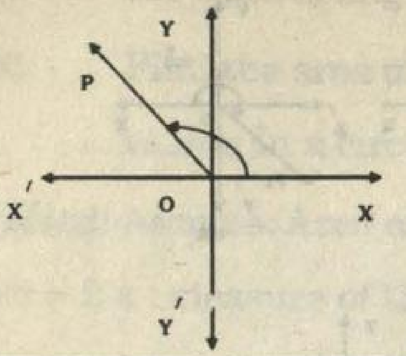


Fig. 8.6

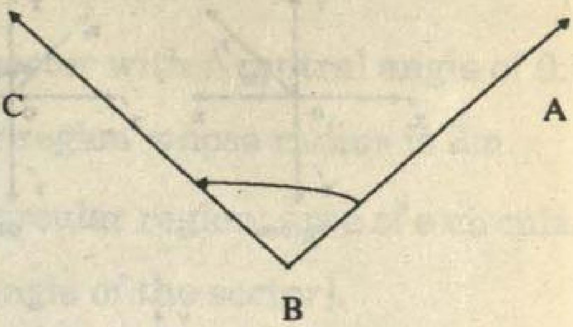


Fig. 8.7

Definitions:

The initial position $\overrightarrow{OX}$ of the revolving ray is called the *initial side* of $\angle$ POX and the ray OP is called it *terminal side*

The ray OP is also called the *Radius Vector*.

When one side of an angle lies along $\overrightarrow{OX}$ it is said to be in the *Standard Position*.

If no coordinate axes have been chosen for representing an angle, say $\angle$ ABC, either of its arms (Ref. fig. 8) may be called the initial side. Its other arm will then be the terminal side. Its measure will be the amount of rotation from $\overrightarrow{BA}$ to $\overrightarrow{BC}$ or vice versa.

If the revolving ray, starting from the original position

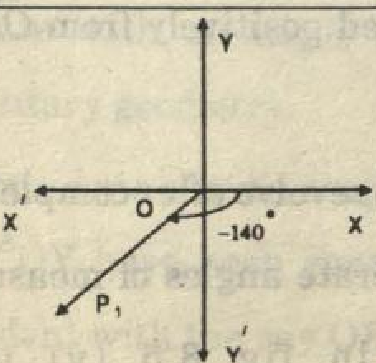


Fig. 8.8

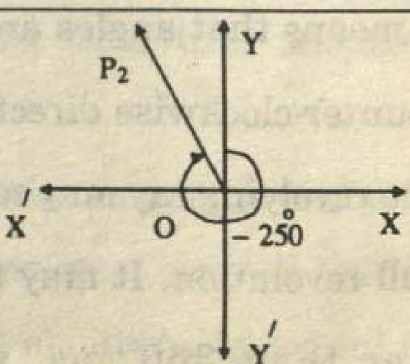


Fig. 8.9

$\overrightarrow{OX}$, revolves in the clockwise direction, it will generate angles of negative measures. For instance:-

In fig (8.8) $m \angle P_1OX = -140^\circ$

In fig (8.9) $m \angle P_2OX = -250^\circ$

Clearly negative angles can also be of any magnitude.

8.4 Trigonometric Functions of any angle

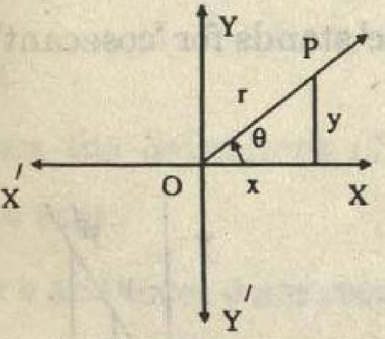


Fig. 8.10

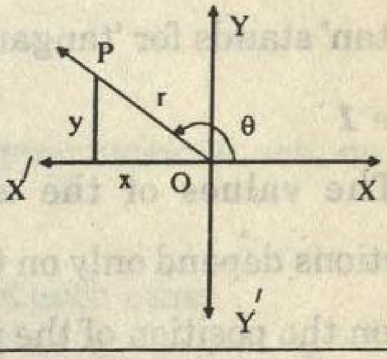


Fig. 8.11

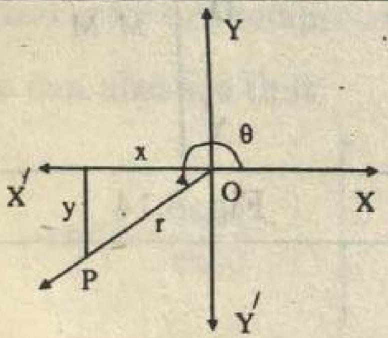


Fig. 8.12

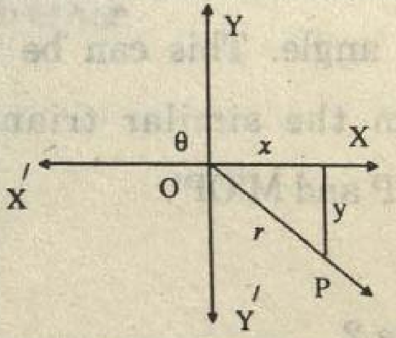


Fig. 8.13

Let POX be any angle in the standard position whose terminal side OP may lie in any one of the four quadrants.

Let (x, y) be the co-ordinates of any point P on the terminal side of the angle.

Let $m \angle POX = \theta$, where θ may be in degrees or in radians.

Let OP = r, (OP = $m\,\overline{OP}$). Then

$$\sin\theta = \frac{y}{r}, \qquad \cot\theta = \frac{x}{y}$$

$$\cos\theta = \frac{x}{r}, \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x}, \qquad \text{cosec}\,\theta = \frac{r}{y}$$

'sin' stands for 'sine' ; 'cot' stands for 'cotangant'

'cos' stands for 'cosine' ; 'sec' stands for 'secant'.

'tan' stands for 'tangant' ; 'cosec' stands for 'cosecant'.

Note 1

The values of the above functions depend only on θ and not on the position of the point P taken on the terminal side of the angle. This can be seen from the similar triangles MOP and M′OP′.

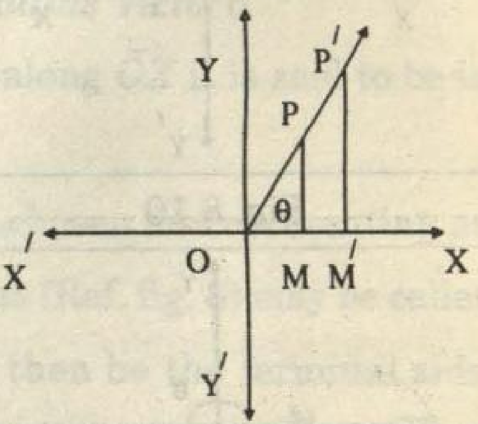


Fig. 8.14

Note 2

It should be noted that sine θ etc are actually functions of θ, for corresponding to any value of θ (when regarded as a variable), there is one and only one value of each of sin θ, cos θ etc.

If sin θ = t, then θ and t are both variables and the sin function may be formally written as the set.

$\{(\theta, t) \mid t = \sin\theta\}$

Similarly, the other functions may be written as sets. sin θ, cos θ etc are called *Trigonometric Functions*.

Note 3

The values of the trigonometric functions for a particular value of θ arc called ***trigonometric ratios***. Thus we may speak of trigonometric ratios (and not trigonometric functions) of 30°, 45°, 210°, – 450° etc.

Note 4

From the definitions of the trigonometric functions we observe that:

sin θ and cosec θ are reciprocals of each other.

Similarly cos θ and sec θ are reciprocals of each other and tan θ and cot θ are reciprocals of each other.

We can also see that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \left[\because \quad \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x}\right]$$

$$\text{and } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

8.5 Fundamental Identities

In whichever quadrant the terminal side of the angle may be, by the Pythagoras Theorem, we have

$|OM|^2 + |MP|^2 = OP^2$ [See figures 8.10 to 8.13]

Where $|OM|$ and $|MP|$ are absolute values of OM and MP

The last equation gives

$$x^2 + y^2 = r^2 \quad \ldots \quad (1)$$

[Recall that $|a|^2 = a^2$]

Dividing equation (1) throughout by r^2, we get

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

i.e., $(\cos\theta)^2 + (\sin\theta)^2 = 1$

or $\sin^2\theta + \cos^2\theta = 1 \quad \ldots \quad (2)$

[writing $\sin^2\theta$ for $(\sin\theta)^2$ and $\cos^2\theta$ for $(cos\theta)^2$]

Again dividing (1) by x^2 we get

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

or $1 + \tan^2\theta = \sec^2\theta \quad \ldots \quad (3)$

Finally dividing (1) by y^2 we get

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

or $\cot^2\theta + 1 = \text{cosec}^2\theta$

or $1 + \cot^2\theta = \text{cosec}^2\theta \quad \ldots \quad (4)$

(2), (3) and (4) are *fundamental indentities* of the trigonometric functions.

[Recall that an identity is an equation which is satisfied by all the permissible values of the variable or variables involved therein.]

Example 5

Prove that:

$$\cos\theta + \tan\theta \sin\theta = \sec\theta \text{ (for all } \theta)$$

Solution

In such questions it is often useful to convert each side of the given equation into sines and cosines.

$$\begin{aligned}\text{L.H.S.} &= \cos\theta + \tan\theta \cdot \sin\theta \\ &= \cos\theta + \frac{\sin\theta}{\cos\theta} \cdot \sin\theta \\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} \\ &= \frac{1}{\cos\theta} \\ &= \sec\theta = \text{R.H.S.}\end{aligned}$$

Example 6

Prove that: $(\sec\theta - \tan\theta)^2 = \dfrac{1 - \sin\theta}{1 + \sin\theta}$ for all θ

Solution

$$\begin{aligned}\text{L.H.S} &= (\sec\theta - \tan\theta)^2 \\ &= \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2 \\ &= \left(\frac{1 - \sin\theta}{\cos\theta}\right)^2 \\ &= \frac{(1 - \sin\theta)^2}{\cos^2\theta}\end{aligned}$$

$$= \frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$

$$= \frac{1-\sin\theta}{1+\sin\theta}$$

$$= \text{R.H.S}$$

8.6 Signs of The Values of Trigonometric Functions in The Four Quadrants

When the terminal side of an angle in standard position lies in a particular quadrant we say that *the angle lies in that quadrant.*

Thus in figures (8.10) to (8.13) above the angle respectively lies in the first, second, third and fourth quadrants.

It is conventional to treat OP = r always positive. Signs of x, y, the coordinates of P are determined in the usual manner. Thus the signs of x, y and r in the various quadrants are as follows: [See figures 8.13]

Quadrant	I	II	III	IV
Sign of x	+	−	−	+
Sign of y	+	+	−	−
Sign of r	+	+	+	+

Therefore, signs of the values of sin θ, cos θ, tan θ in the four quadrants are as shown in fig 8.15 (in which S, C and T respectively stand for

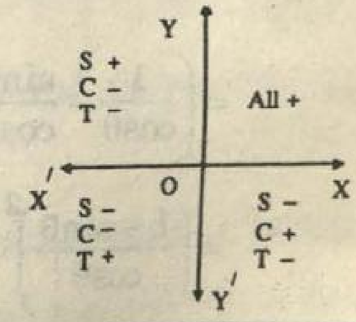


Fig. 8.15

sinθ, cosθ and tanθ. Their reciprocals cosecθ, secθ and cotθ will respectively have the same signs.

8.7 Given the Value of one Trigonometric Ratio, of an Angle to Find the Values of other Trigonometric Ratios

Example 7

If $\sin\theta = \frac{2}{3}$ and the terminal side of the angle lies in the second quadrant, find the values of the remaining trigonometric ratios of θ

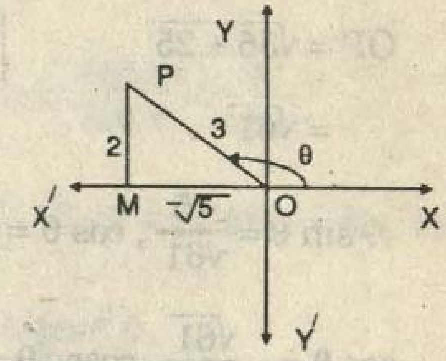


Fig. 8.16

Solution

In this case

$$MP = 2, \qquad OP = 3$$

$$OM = -\sqrt{9-4} = -\sqrt{5}$$

$$\therefore \cos\theta = \frac{-\sqrt{5}}{3}, \ \tan\theta = \frac{-2}{\sqrt{5}},$$

$$\cot\theta = \frac{-\sqrt{5}}{2}, \ \sec\theta = \frac{-3}{\sqrt{5}}, \ \operatorname{cosec}\theta = \frac{3}{2}$$

Example 8

If $\tan\theta = \frac{5}{6}$ and the terminal side of the angle is in the third quadrant, find the remaining trigonometric ratios of θ.

Solution

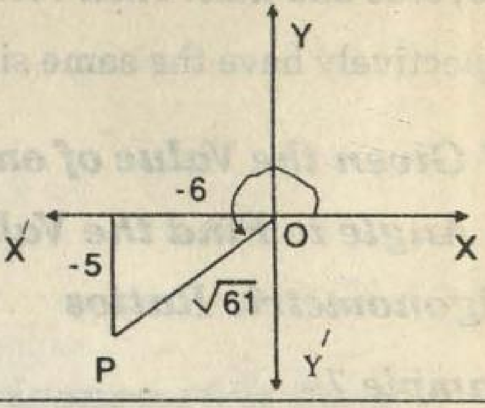


Fig. 8.17

In this case

$OM = -6,$

$MP = -5$

So that $\tan\theta = \frac{-5}{-6} = \frac{5}{6}$

$OP = \sqrt{36 + 25}$

$= \sqrt{61}$

$\therefore \sin\theta = \frac{-5}{\sqrt{61}}, \cos\theta = \frac{-6}{\sqrt{61}}, \cot\theta = \frac{-6}{-5} = \frac{6}{5}$

$\sec\theta = -\frac{\sqrt{61}}{6}, \operatorname{cosec}\theta = -\frac{\sqrt{61}}{5}$

Exercise 8.2

1. Where will the terminal side lie for the given measures of angles?

(i) $\frac{3\pi}{2}$ (ii) $-\pi$ (iii) 5π

(iv) -7π (v) $-\frac{11\pi}{2}$ (vi) 170°

(vii) -330° (viii) -500° (ix) 720°

(x) -1080° (xi) -720° (xii) -900°.

2. Find the signs of the following ratios:-

(i) $\sin 160^\circ$ (ii) $\cos 160^\circ$ (iii) $\tan 200^\circ$

(iv) $\sin 262^\circ$ (v) $\sin(-40^\circ)$ (vi) $\cot(-130^\circ)$

(vii) $\sin 210^\circ$ (viii) $\tan 210^\circ$ (ix) $\cos 470^\circ$

(x) $\sin 470^\circ$ (xi) $\sin(-470^\circ)$ (xii) $\tan(-470^\circ)$

(xiii) $\sin 1030^\circ$ (xiv) $\cos 1000$ (xv) $\tan(-1000^\circ)$.

[***Hint***: Neglect 360°, -360° and their multiples].

Prove the following identities:-

3. $\cot\theta \sin\theta = \cos\theta$

4. $\dfrac{\sec\theta}{\operatorname{cosec}\theta} = \tan\theta$ 5. $\dfrac{\tan x}{\sin x} = \dfrac{1}{\cos x}$

6. $\dfrac{\sin x}{\operatorname{cosec} x} + \dfrac{\cos x}{\sec x} = 1$ 7. $(1+\sin\theta)(1-\sin\theta) = \dfrac{1}{\sec^2\theta}$

8. $(\sec\theta - 1)(\sec\theta + 1) = \tan^2\theta$

9. $(1-\sin^2\theta)(1+\tan^2\theta) = 1$

10. $(1-\cos^2\theta)(1+\cot^2\theta) = 1$

11. $\sin x(\operatorname{cosec} x - \sin x) = \cos^2 x$

12. $\cos x(\tan x + \cot x) = \operatorname{cosec} x$

13. $1 - 2\sin^2\theta = 2\cos^2\theta - 1$

14. $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta$

15. $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

16. $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$

17. $(\operatorname{cosec}\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

18. $\dfrac{\cot^2\theta - 1}{\cot^2\theta + 1} = 2\cos^2\theta - 1$

19. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

20. $\dfrac{\cos^3 t - \sin^3 t}{\cos t - \sin t} = 1 + \sin t \cdot \cos t$

21. $\sec^2 t + \tan^2 t = (1 - \sin^4 t)\sec^4 t$

22. $\dfrac{\sec x - \cos x}{1 + \cos x} = \sec x - 1$

23. $\dfrac{\sin x + \cos x}{\tan^2 x - 1} = \dfrac{\cos^2 x}{\sin x - \cos x}$

24. If $\frac{\pi}{2} < \theta < \pi$, find all the trignomatric ratios in terms of $\cos\theta$

25. If $\frac{\pi}{2} < \theta < \pi$ and $\sin\theta = \frac{4}{5}$, find the remaining five trignometric ratios for the same value of θ.

Find the remaining trigonometric ratios in each of the following cases (Q. 26 – 29).

26. $\cot\theta = 4/3$ and the terminal side is not in the first quadrant.

27. $\sin\theta = 2/3$ and the terminal side is not in the first quadrant.

28. $\tan\theta = -1/3$ and the terminal side is in the second quadrant.

29. $\cos\theta = \dfrac{-\sqrt{3}}{2}$ and the terminal side is in the third quadrant.

30. In which quadrant does the terminal side lie if

(i) $\sin\theta < 0$ and $\cos\theta > 0$,

(ii) $\sin\theta > 0$ and $\cot\theta > 0$,

(iii) $\sin\theta < 0$ and $\tan\theta > 0$,

(iv) $\tan\theta < 0$ and $\sec\theta > 0$,

(v) $\cos\theta$ and $\text{cosec}\,\theta$ are both negative,

(vi) $\cos\theta$ and $\tan\theta$ are both negative.

8.8 Trigonometric Ratios of Some Special Angles

From our previous knowledge we already know the trigonometric ratios of 30^O, 45^O and 60^O. We now find the trigonometric ratios of 90^O, 0^O, 180^O and 270^O

8.8.1 Trigonometric Ratios of 0°.

In this case $\overrightarrow{OP}$ coincides with $\overrightarrow{OX}$ and P coincides with M.

$\therefore$ OM = OP

i.e $x = r$, and $y = 0$

$$\therefore \sin 0^o = \frac{0}{r} = 0$$

$$\cos 0^o = \frac{x}{r} = \frac{r}{r} = 1$$

$$\tan 0^o = \frac{y}{x} = \frac{0}{x} = 0,$$

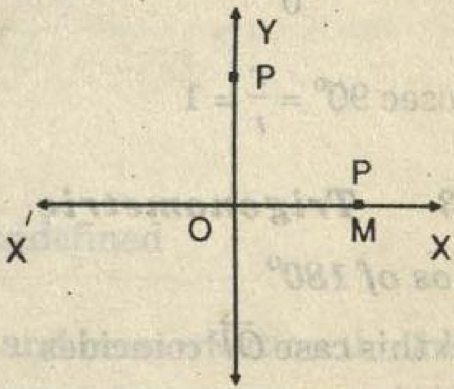


Fig. 8.18

$$\cot 0^o = \frac{1}{0} =$$

undefined.

$\sec 0^o = \frac{r}{r} = 1$, $\text{cosec}\ 0^o = \frac{1}{0}$ = undefined.

8.8.2 Trigonometric Ratios of 90^o

In this case $\overrightarrow{OP}$ coincides with $\overrightarrow{OY}$ and M coincides with O.

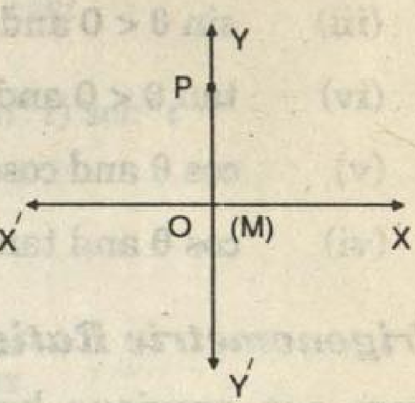


Fig. 8.19

$$\therefore\ x = 0\ , y = r$$

$$\therefore\ \sin 90^o = \frac{y}{r} = \frac{r}{r} = 1$$

$$\cos 90^o = \frac{x}{r} = \frac{0}{r} = 0$$

$$\tan 90^o = \frac{y}{x} = \frac{y}{0} = \text{undefined,}$$

$$\cot 90^o = \frac{x}{y} = \frac{0}{y} = 0$$

$$\sec 90^o = \frac{1}{0} = \text{undefined}$$

$$\text{cosec } 90^o = \frac{r}{r} = 1$$

8.8.3 Trigonometric Ratios of 180^o

In this case $\overrightarrow{OP}$ coincides with $\overrightarrow{OX'}$ and M coincides with P

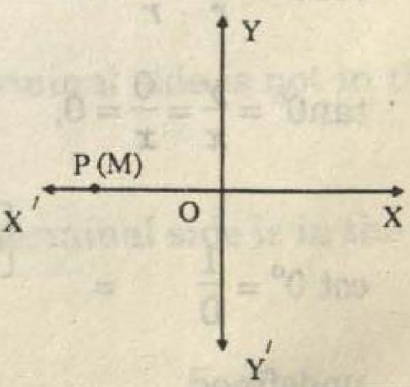


Fig. 8.20

$$\therefore\ y = 0$$

and $-x = r$

($\because$ OM is –ve and OP is +ve)

$$\therefore\ \sin 180^\circ = \frac{y}{r} = \frac{0}{r} = 0, \qquad \cos 180^\circ = \frac{x}{r} = \frac{-r}{r} = -1$$

$$\tan 180^\circ = \frac{y}{x} = \frac{0}{x} = 0.$$

Write values of the other ratios your selves.

8.8.4 Trigonometric Ratios of 270°

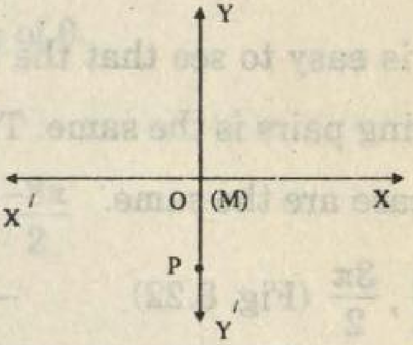


Fig. 8.21

In this case $\overrightarrow{OP}$ coincides with $\overrightarrow{OY'}$ and M falls on O

$$\therefore\ x = 0$$

$$-y = r$$

($\because$ MP is – ve and OP is + ve).

$$\therefore\ \sin 270^\circ = \frac{y}{r} = \frac{-r}{r} = -1$$

$$\cos 270^\circ = \frac{x}{r} = \frac{0}{r} = 0$$

$$\tan 270^\circ = \frac{y}{x} = \frac{y}{0} = \text{undefined}$$

For the sake of convenience and memorization the following table is given for the trigonometric ratios discussed above.

θ	0°	30°	45°	60°	90°	180°	270°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	- 1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	- 1	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	×	0	×

NOTE: Values of trigonometric ratios of acute angles other than those given above can be found from the tables of these ratios.

8.9 Trigonometric Ratios of Negative Multiples of π.

It is easy to see that the terminal side $\overrightarrow{OP}$ in each of the following pairs is the same. Therefore, trigonometric ratios in each case are the same.

$-\frac{\pi}{2}, \frac{3\pi}{2}$ (Fig. 8.22) $-\pi, \pi$ (Fig. 8.23)

$-\frac{3\pi}{2}, \frac{\pi}{2}$ (Fig. 8.24)

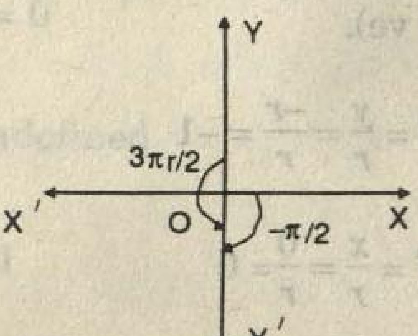


Fig. 8.22

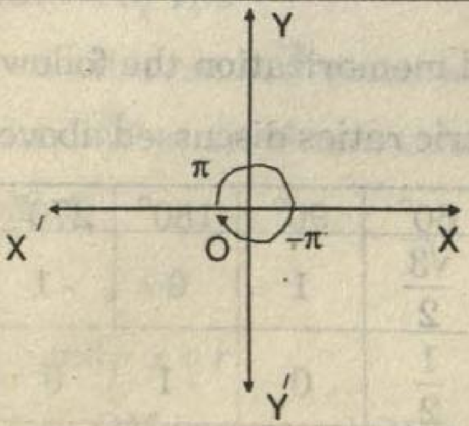


Fig. 8.23

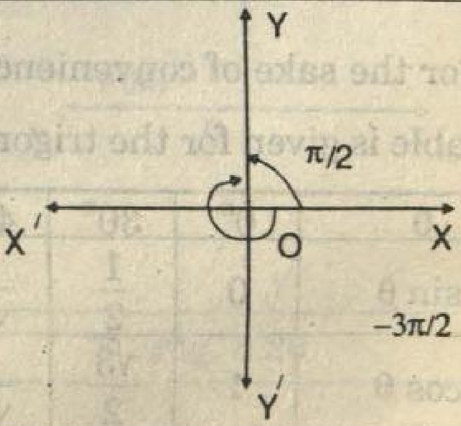


Fig. 8.24

8.10 Trigonometric Ratios of Measures of the Form $\theta \pm 2k\pi$ *(k* is a positive integer)

Since the revolving ray $\overrightarrow{OP}$ returns to the initial position after having completed every full revolution, trigonometric ratios of $\theta \pm 2k\pi$ are the same as of θ.

Example 1

Find trigonometric ratios of $\frac{-3\pi}{2}$

Solution

The terminal side coincides with $\overrightarrow{OY}$

$\therefore$ trigonometric ratios are the same as of $\pi/2$

$$\therefore \sin\left(\frac{-3\pi}{2}\right) = \sin\frac{\pi}{2} = 1,\ \cos\left(\frac{-3\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

The remaining ratios can now be written down with the help of these two ratios.

Example 2

Find trigonometric ratios of 9π.

Solution

$$9\pi = 4 \,.\, 2\pi + \pi$$

$\therefore$ trigonometric ratios of 9π are the same as those of π.

$$\therefore \sin 9\pi = \sin \pi = 0$$

$$\cos 9\pi = \cos \pi = -1, \text{ etc.}$$

Example 3

Find the trigonometric ratios of $\frac{-13}{2}\pi$.

Solution

$$\frac{-13\pi}{2} = -\left(6 + \frac{1}{2}\right)$$

$$\therefore \quad \frac{-13}{2}\pi = -6\pi - \frac{\pi}{2} = -3 \cdot 2\pi - \frac{\pi}{2}$$

$\therefore$ trigonometric ratios of $\frac{-13}{2}\pi$ are the same as of $-\frac{\pi}{2}$ i.e., of $\frac{3\pi}{2}$.

$$\therefore \sin\left(\frac{-13}{2}\pi\right) = \sin\frac{3\pi}{2} = -1$$

$$\cos\left(\frac{-13}{2}\pi\right) = \sin\frac{3\pi}{2} = 0, \text{ etc.}$$

Example 4

Find trigonometric ratios of -1980°.

Solution

$$-1980 = -(5 \times 360 + 180)$$

$\therefore$ trigonometric ratios of -1980° are the same as those of -180° i.e. the same as those of 180°.

$$\therefore \sin(-1980^\circ) = \sin 180^\circ = 0$$

$$\cos(-1980^\circ) = \cos 180^\circ = -1$$

$$\tan(-1980^\circ) = \tan 180^\circ = 0$$

The rest of the ratios can be written which are the reciprocals of these three ratios.

Exercise 8.3

1. Evaluate each of the following expressions:-

(i) $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ$.

(ii) $\cos 30^\circ \cos 60^\circ - \sin 30 \sin 60^\circ$.

(iii) $\dfrac{\tan 60^o - \tan 30^o}{1 + \tan 60^o \tan 30^o}$.

2. Verify the following equalities:-

(i) $\sin 90^\circ = 2 \sin 45^\circ \cos 45^\circ$

(ii) $\cos 90^\circ = 1 - 2 \sin^2 45^\circ$

(iii) $\cos 90^\circ - \cos 30^\circ = - 2 \sin 60^\circ \sin 30^\circ$

Find trigonometric ratios of

3. $-\pi$
4. -3π
5. $-\dfrac{5\pi}{2}$
6. $\dfrac{-9\pi}{2}$
7. -15π
8. -1530°
9. -2430°
10. $\dfrac{235\pi}{2}$
11. $\dfrac{407\pi}{2}$

Chapter 9

Trigonometric Identities

9.1 Distance Formula

Before discussing trigonometric indentics, we prove the distance formula of co-ordinate Geometry. This formula will be used in the next section.

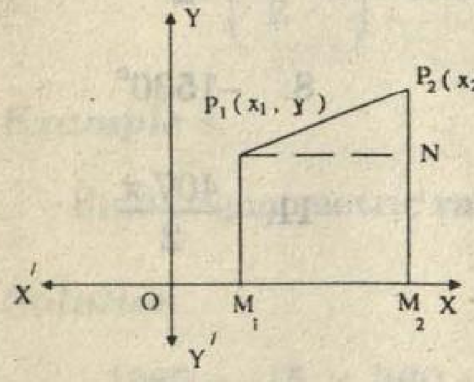

Y
P₁(x₁, y₁)
P₂(x₂, y₂)
N
X'
O
M₁
M₂
X
Y'

Fig. 9.1

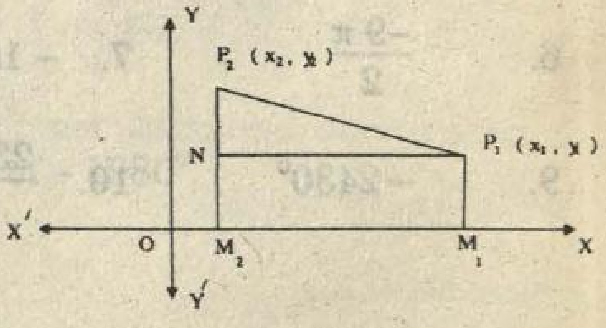

Y
P₂ (x₂, y₂)
N
P₁ (x₁, y₁)
X'
O
M₂
M₁
X
Y'

Fig. 9.2

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points in the co-ordinate plane. We have to find the distance '*d*' between them.

In fig 9.1

$$P_1N = M_1M_2 = OM_2 - OM_1$$

$$= x_2 - x_1$$

$$NP_2 = M_2P_2 - M_2N$$

$$M_2P_2 - M_1P_1 = y_2 - y_1$$

In fig 9.2 with the coordinates of P_2 are greater than those of P_1. But it can be easily seen that the above formula is valid for all the relative positions of P_1 and P_2 for instance, in fig 9.2

$$d^2 = (x_1 - x_2)^2 + (y_2 - y_1)^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Similarly in fig 9.3

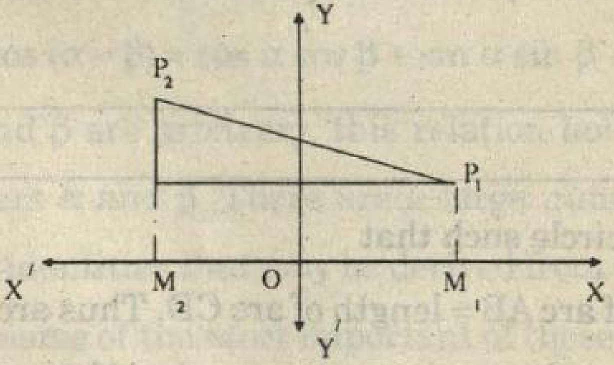


Fig. 9.3

$$d^2 = [x_1 + (-x_2)^2] + (y_2 - y_1)^2$$ (In this case x_2 is negative)

$$= (x_1 - x_2)^2 + (y_2 - y_1)^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Cor

The distance r of the point (x_1, y_1) from the origin is given by

$r = \sqrt{x_1^2 + y_1^2}$ (Putting $x_2 = 0 = y_2$ in the formula)

9.2 A Fundamental Law

Let us take a unit circle with Centre O. Take points A, B,

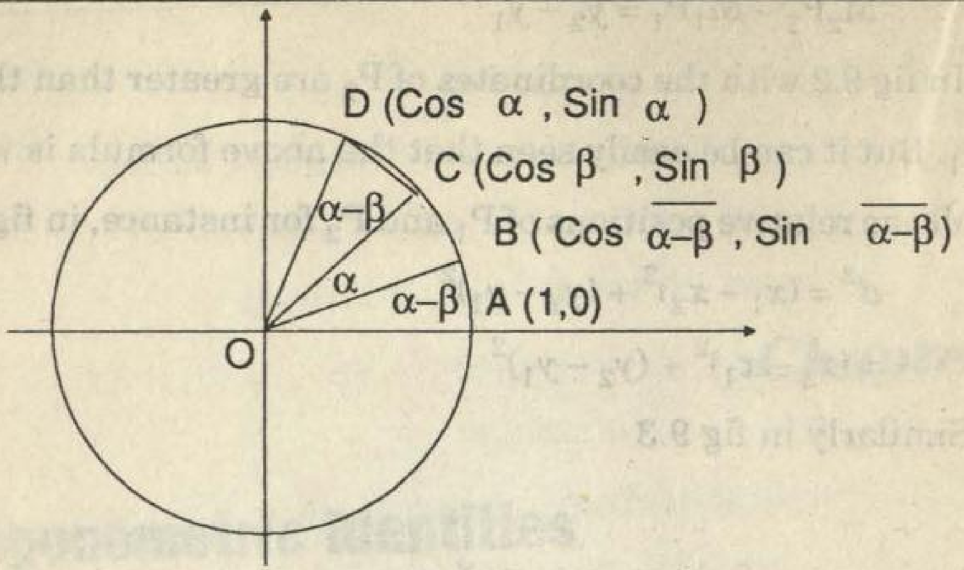

D (Cos α , Sin α)
C (Cos β , Sin β)
B (Cos $\overline{\alpha-\beta}$, Sin $\overline{\alpha-\beta}$)
A (1,0)
O
α−β
α
α−β

Fig. 9.4

C, D on the circle such that

Length of arc AB = length of arc CD. Thus arc AB ≅ arc CD. We know from elementary geometry that if two arcs in a circle are congruent, then their chords are congruent, i.e., they are of equal length.

Thus chord AB ≅ Chord CD, i.e.,

$$|AB| = |CD|$$

Now let us suppose

$$m\angle AOD = \alpha$$

$$m\angle AOC = \beta$$

$$\therefore m\angle COD = \alpha - \beta$$

and since $|AB| = |CD|$

$$\therefore m\angle COD = m\angle AOB$$

Hence $m \angle AOB = \alpha - \beta$

Therefore, the co-ordinates of the points A, B, C, D are respectively $(1,0)$, $(\cos \overline{\alpha - \beta}, \sin \overline{\alpha - \beta})$, $(\cos \beta, \sin \beta)$, $(\cos \alpha, \sin \alpha)$

Since $|AB|^2 = |CD|^2$, we have by the distance formula.

$$(1 - \cos \overline{\alpha - \beta})^2 + (0 - \sin \overline{\alpha - \beta})^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$$

$$2 - 2 \cos (\alpha - \beta) = 1 + 1 - 2 (\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$

$$\text{i.e} \quad \cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad \ldots \quad (1)$$

Since α and β are arbitrary, this relation holds for every pair of numbers α and β. There are a large number of basic trigonometric identities that may be derived from (1). We shall next consider some of the most important of these. We refer to (1) as *Fundamental Law.*

9.3 Deductions from the Fundamental Law

$$\text{Cos} (-\beta) = \text{Cos} \beta \quad \ldots \quad (2)$$

Put $\alpha = 0$ in (1)

Then $\cos (-\beta) = \cos 0 \cos \beta + \sin 0 \sin \beta$

$= (1) \cos \beta + (0) \sin \beta$

$\cos (-\beta) = \cos \beta$ as $\cos 0 = 1$ and $\sin 0 = 0$

$$\cos \left(\frac{\pi}{2} - \beta\right) = \sin \beta \quad \ldots \quad (3)$$

Put $\alpha = \frac{\pi}{2}$ in (1)

Then $\cos \left(\frac{\pi}{2} - \beta\right) = \cos \frac{\pi}{2} \cos \beta + \sin \frac{\pi}{2} \sin \beta$

$$= (0)\cos\beta + (1)\sin\beta$$

$$= \sin\beta$$

$$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha \quad \ldots \quad (4)$$

Put $\beta = -\frac{\pi}{2}$ in (1)

$$\cos\left(\alpha + \frac{\pi}{2}\right) = \cos\alpha\cos\left(\frac{-\pi}{2}\right) + \sin\alpha\sin\left(\frac{-\pi}{2}\right)$$

$$= \cos\alpha\cos\frac{\pi}{2} + (\sin\alpha)(-1)$$

$$= (\cos\alpha)\,0 - \sin\alpha$$

$$= -\sin\alpha$$

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha \quad \ldots \quad (5)$$

Put $\beta = \frac{\pi}{2} + \alpha$ in (3)

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2} - \alpha\right) = \sin\left(\frac{\pi}{2} + \alpha\right)$$

$$\cos(-\alpha) = \sin\left(\frac{\pi}{2} + \alpha\right)$$

i.e $\sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$ by 2

$$\text{Sin}(-\beta) = -\sin\beta \quad \ldots \quad (6)$$

Put $\alpha = -\beta$ in (4)

Then $\cos\left(\frac{\pi}{2} - \beta\right) = -\sin(-\beta)$

But by 3, $\cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$

Hence $\sin\beta = -\sin(-\beta)$

i.e., $\sin(-\beta) = -\sin\beta$

$\tan(-\beta) = -\tan\beta \quad \ldots \quad (7)$

$$\tan(-\beta) = \frac{\sin(-\beta)}{\cos(-\beta)}$$

We now use 2 and 6

$$\tan(-\beta) = \frac{-\sin\beta}{\cos\beta}$$

$$= -\tan\beta$$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad \ldots(8)$

substituting β for $(-\beta)$ in (1), we get

$$\cos[\alpha-(-\beta)] = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

or $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \ldots (9)$

Replace α by $\left(\frac{\pi}{2}+\alpha\right)$ in (8)

Then

$$\cos\left[\left(\frac{\pi}{2}+\alpha\right)+\beta\right] = \cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta - \sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$$

But $\cos\left[\frac{\pi}{2}+(\alpha+\beta)\right] = -\sin(\alpha+\beta)$ by (4)

$\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha \qquad$ by (4)

and $\sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha \qquad$ by (5)

Thus $\cos\left[\frac{\pi}{2}+(\alpha+\beta)\right] = \cos\left(\frac{\pi}{2}+\alpha\right)\cos\beta - \sin\left(\frac{\pi}{2}+\alpha\right)\sin\beta$

reduces to

$$-\sin(\alpha+\beta) = -\sin\alpha\cos\beta - \cos\alpha\sin\beta$$

or $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad \ldots(10)$$

Replace β by $-\beta$ in (9)

Then $$\sin(\alpha-\beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(\pi-\beta) = \sin\beta \quad \ldots \quad (11)$$

$$\sin(\pi-\beta) = \sin\pi\cos\beta - \cos\pi\sin\beta$$

$$= 0\cos\beta - (-1)\sin\beta$$

$$= \sin\beta$$

$$\cos(\pi-\beta) = -\cos\beta \quad \ldots \quad (12)$$

$$\cos(\pi-\beta) = \cos\pi\cos\beta + \sin\pi\sin\beta$$

$$= (-1)\cos\beta + 0\sin\beta$$

$$= -\cos\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \quad \ldots \quad (13)$$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

(Dividing numerator and denominator by $\cos\alpha\cos\beta$)

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \quad \ldots \quad (14)$$

We replace β by (– β) and use (7)

Hence $$\tan (\alpha - \beta) = \frac{\tan \alpha + \tan (-\beta)}{1 + \tan \alpha \tan (-\beta)}$$

$$= \frac{\tan \alpha - \tan \beta}{1 - \tan \alpha \tan \beta}$$

Example 1

Find the value of sin 75°

Solution

$$\sin 75^\circ = \sin (45^0 + 30^\circ)$$

$$= \sin 45^0 \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Example 2

Show that cos (270° – θ) = – sin θ

Solution

$$\cos (270^\circ - \theta) = \cos 270 \cos \theta + \sin 270 \sin \theta$$

$$= (0) \cos \theta + (-1) \sin \theta$$

$$= -\sin \theta$$

Example 3

Prove that $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

Solution

$$\begin{aligned}\cos\theta &= \cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right)\\ &= \cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\cdot\sin\frac{\theta}{2}\\ &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\\ &= \cos^2\frac{\theta}{2} - \left(1-\cos^2\frac{\theta}{2}\right)\\ &= 2\cos^2\frac{\theta}{2} - 1\end{aligned}$$

Example 4

Given $\sin\alpha = \frac{12}{13}$ and $\tan\beta = \frac{4}{3}$, with α and β in the first quadrant. Find in what quadrant $(\alpha+\beta)$ lies.

Solution

Since α and β are both in the first quadrant. Tangent is positive in the first quadrant and negative in the second, therefore, $\tan(\alpha+\beta)$ will decide the quadrant in which $(\alpha+\beta)$ lies.

If $\sin\alpha = \frac{12}{13}$, then $\tan\alpha = \frac{12}{5}$

Now $\quad \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$$= \frac{\frac{12}{5} + \frac{4}{3}}{1 - \frac{12}{5} \cdot \frac{4}{3}}$$

$$= \frac{\frac{36+20}{15}}{\frac{15-48}{15}}$$

$$\tan(\alpha + \beta) \qquad = -\frac{56}{33}$$

Since $\tan(\alpha + \beta)$ is negative, it follows that $(\alpha + \beta)$ is in the second quadrant.

Example 5

Express sin 196° as a function of a positive degree measure less than 45°.

Solution

$$\sin 196^\circ = \sin(\pi + 16^\circ)$$

$$= -\sin 16^\circ$$

Example 6

Express $4 \sin \theta + 3 \cos \theta$ in the form $r \sin(\theta + \varphi)$ where terminal ray of ϕ is in the first quadrant.

Solution

By multiplying and dividing the expression by $\sqrt{(4)^2 + (3)^2} = 5$ we have

$$5\left[\frac{4}{5} \sin \theta + \frac{3}{5} \cos \theta\right]$$

$$= 5\left[(\sin\theta)\left(\frac{4}{5}\right) + (\cos\theta)\left(\frac{3}{5}\right)\right]$$

Let $\cos\varphi = \frac{4}{5}$, we find that $\sin\varphi = \frac{3}{5}$; thus the above expression becomes.

$$5\,[\sin\theta\cos\varphi + \cos\theta\sin\varphi]$$

$$= 5\sin(\theta + \varphi) \text{ where } \sin\varphi = \frac{3}{5} \text{ and } \cos\varphi = \frac{4}{5}, \ r = 5$$

Exercise 9.1

1. Prove that

(i) $\sin(180^\circ - \theta) = \sin\theta$

(ii) $\cos(180^\circ + \theta) = -\cos\theta$

(iii) $\tan(90^\circ + \theta) = -\cot\theta$

(iv) $\tan(270^\circ + \theta) = -\cot\theta$

(*v*) $\cot(360^\circ + \theta) = \cot\theta$

(vi) $\tan(180^\circ + \theta) = \tan\theta$

(vii) $\sin(360^\circ - \theta) = -\sin\theta$

(viii) $\sin(\theta + 270^\circ) = -\cos\theta$

(ix) $\cos(270^\circ + \theta) = \sin\theta$

2. Show that

(i) $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$

(ii) $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$

(iii) $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$

(iv) $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta$

3. Show that

(i) $\cos\beta = \cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2} = 2\cos^2\frac{\beta}{2} - 1 = 1 - 2\sin^2\frac{\beta}{2}$

(ii) $\sin\alpha = 2\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}$

4. Show that

(i) $\cos(\alpha+\beta).\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$

(ii) $\sin(\alpha+\beta).\sin(\alpha-\beta) = \cos^2\beta - \cos^2\alpha$

5. prove that

(i) $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

(ii) $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

6. If $\tan\alpha = \frac{3}{4}$, $\cos\beta = \frac{5}{13}$ neither $R(\alpha)$ nor $R(\beta)$ is in the first quadrant, find

(i) $\sin(\alpha+\beta)$ (ii) $\cos(\alpha+\beta)$

7. Prove that

(i) $\tan(45^\circ - \theta) = \frac{1-\tan\theta}{1+\tan\theta}$

(ii) $\tan(45^\circ + \theta) = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

(iii) $\frac{\tan(\alpha+\beta)}{\cot(\alpha-\beta)} = \frac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha\cdot\tan^2\beta}$

(iv) $\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

(v) $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

8. If $\cos\alpha = \dfrac{1}{7}$, $\cos\beta = \dfrac{13}{14}$, where the terminal ray of α and β are in the first quadrant, then prove that $|\alpha - \beta| = 60^\circ$

9. Find the value of

(i) $\sin 15^\circ$ (ii) $\cos 15^\circ$ (iii) $\tan 15^\circ$

by taking $15^\circ = 60^\circ - 45^\circ$

10. If $\sin\alpha = \dfrac{4}{5}$ and $\sin\beta = \dfrac{12}{13}$, where neither terminal ray of α nor terminal ray of β is in the first quadrant, find

(i) $\sin(\alpha+\beta)$ (ii) $\sin(\alpha-\beta)$

(iii) $\cos(\alpha+\beta)$ (iv) $\cos(\alpha-\beta)$

11. Find the value of (i) $\sin 2\theta$, (ii) $\cos 2\theta$ (iii) $\tan 2\theta$, given

(a) $\sin\theta = \dfrac{3}{5}$,terminal ray of θ is in the first quadrant

(*b*) $\tan\theta = -\dfrac{1}{5}$ terminal ray of θ is in the second quadrant.

12. Express the following in the form $r\sin(\theta+\varphi)$ where terminal ray of θ and ϕ are in the first quadrant.

(i) $12\sin\theta + 5\cos\theta$

(ii) $4\sin\theta + 7\cos\theta$

(iii) $5 \sin \theta - 4 \cos \theta$

(iv) $13 \sin \theta + 5 \cos \theta$

(*v*) $3 \sin \theta - 4 \cos \theta$

(vi) $\sin \theta + \cos \theta$

13. Express each of the following as function of a positive degree measure less than 45°

(i) $\sin 190^\circ$ (ii) $\cos 253^\circ$

(iii) $\tan 295^\circ$ (iv) $\sin 248^\circ 20'$

(v) $\cos 200^\circ 40'$ (vi) $\tan 135^\circ 30'$

14. Prove that $\dfrac{\sin \alpha}{\sec 4\alpha} + \dfrac{\cos \alpha}{\operatorname{cosec} 4\alpha} = \sin 5\alpha$

15. Prove that $\dfrac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \dfrac{\sin (\alpha + \beta)}{\sin (\alpha - \beta)}$

16. Find the value of (i) $\sin 105^\circ$ (ii) $\cos 105^\circ$ (iii) $\tan 105^\circ$ by taking $105^\circ = 60^\circ + 45^\circ$

9.4 Double angle identities

We know that $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ ... (A)

Suppose $\beta = \alpha$ and substitute α for β in (A)

then $\sin (\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$

or $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ ... (15)

Also $\cos (\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha$

$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$ ... (16)

In (16) above, substitute $1 - \sin^2 \alpha$ for $\cos^2 \alpha$. This gives

$\cos 2\alpha = 1 - 2 \sin^2 \alpha$ ... (17)

or substitute $1 - \cos^2 \alpha$ for $\sin^2 \alpha$ in (16)

Then

$$\cos 2\alpha = 2\,\mathrm{Cos}^2 \alpha - 1 \quad \ldots \quad (18)$$

and $\tan(\alpha + \alpha) = \dfrac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$, where $\tan\alpha \neq \pm 1$

or

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \quad \ldots \quad (19)$$

9.5 Half angle identities

By identity (17)

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\therefore \sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\text{or } \sin\alpha = \pm\sqrt{\frac{1 - \cos 2\alpha}{2}}$$

Since these identities are true for any value of α, we may substitute $\frac{\theta}{2}$ for α and θ for 2α

Thus

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}} \quad \ldots \quad (20)$$

If θ is an angle that $\frac{\theta}{2}$ lies in the first or the second quadrant we will use the identity (20) with the positive sign; if $\frac{\theta}{2}$ *lies in* the third or fourth quadrant we will use the identity with the negative sign. When using the cosine or tangent identities which follow we will interpret the positive or negative sign in a similar manner according to the quadrant in which $\frac{\theta}{2}$ lies.

It is very important to note that if the terminal ray of θ lies, for example, in the second quadrant then it does not automatically follows that the terminal ray of $\frac{\theta}{2}$ will lie in the first quadrant, for example if,

$\theta = 2\pi + \frac{3\pi}{4}$, then the terminal ray of θ lies in the second quadrant but the corresponding ray of $\frac{\theta}{2} = \pi + \frac{3\pi}{8}$ is in the third quadrant.

Now by (18)

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$$

$$\cos \alpha = \pm\sqrt{\frac{1 + \cos 2\alpha}{2}}$$

Substituting $\frac{\theta}{2}$ for α, θ for 2 α, we get $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$ ... (21)

Now $$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

Using (20) and (21)

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} \quad \ldots \quad (22)$$

Example 1

If $\sin\theta = \frac{4}{5}$ and terminal ray of θ is in the second quadrant, find the value of (i) $\sin 2\theta$, (ii) $\cos\frac{\theta}{2}$

Solution

If $\sin\theta = \frac{4}{5}$ and terminal ray of θ is in the second quadrant then $\cos\theta < 0$

$$\text{so} \quad \cos\theta = -\sqrt{1-\sin^2\theta}$$

$$= -\sqrt{1-\left(\frac{4}{5}\right)^2}$$

$$= -\frac{3}{5}$$

$$\text{Now} \quad \sin 2\theta = 2\sin\theta\cdot\cos\theta$$

$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$$

$$= -\frac{24}{25}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

$$= \sqrt{\frac{1-\frac{3}{5}}{2}} \quad = \frac{1}{\sqrt{5}}$$

Example 2

Compute the value of $\sin\frac{\pi}{12}$ from the function $\frac{\pi}{6}$

Solution

We know that $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ see (20)

$$\therefore \sin\frac{\pi}{12} = \sqrt{\frac{1-\cos\frac{\pi}{6}}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{2}}$$

Example 3

Prove that $\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha} = 2$

Solution

Taking L.H.S.

$$\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha} = \frac{\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha}{\sin\alpha\cos\alpha}$$

$$= \frac{\sin(3\alpha-\alpha)}{\sin\alpha\cos\alpha}$$

$$= \frac{\sin 2\alpha}{\sin\alpha\cos\alpha}$$

$$= \frac{2\sin\alpha\cos\alpha}{\sin\alpha\cos\alpha}$$

$$= 2 \quad = \text{R.H.S.}$$

Example 4

Find $\cos 3\theta$ in terms of $\cos\theta$.

Solution

$$\begin{aligned}\cos 3\theta &= \cos(\theta + 2\theta)\\ &= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta\\ &= \cos\theta\,[2\cos^2 - 1] - \sin\theta\,(2\sin\theta\cos\theta)\\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

Exercise 9.2

1. If $\cos\alpha = \frac{4}{5}$ and terminal ray of α is in the first quadrant, find the value of

(i) $\tan 2\alpha$ (ii) $\sin\frac{\alpha}{2}$

(iii) $\cos\frac{\alpha}{2}$ (iv) $\tan\frac{\alpha}{2}$

2. $\sin\beta = \frac{12}{13}$ and terminal ray of β is in the second quadrant, find the value of

(i) $\sin 2\beta$ (ii) $\sin\frac{\beta}{2}$

(iii) $\cos\frac{\beta}{2}$ (iv) $\tan\frac{\beta}{2}$

3. Given $\tan\theta = -\frac{3}{4}$ and terminal of ray θ is in the second quadrant, find the value of

(i) $\sin\frac{\theta}{2}$ (ii) $\cos\frac{\theta}{2}$

4. Given $\cos\theta = -\frac{1}{5}$, and terminal ray of θ is in the third quadrant, find $\tan\frac{\theta}{2}$

5. Express $\cos^4\theta$ in terms of trigonometric functions of multiples of θ with exponent as unity.

6. Compute the value of (i) $\cos\frac{2\pi}{3}$ (ii) $\sin\frac{2\pi}{3}$ using double angle identities.

Prove the following identities

7. $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$

8. $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$

9. $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$, show that $\cos 2\theta = \frac{1-t^2}{1+t^2}$ where $t = \tan\theta$.

10. $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$

11. $\frac{\sin\alpha}{2} = \sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}$

12. $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$

13. $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

14. $\dfrac{1-\cos 2\alpha}{\sin 2\alpha} = \tan\alpha$

15. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

16. $\tan\alpha \cdot \tan\dfrac{\alpha}{2} = \sec\alpha - 1$

17. $\left(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2}\right)^2 = 1 + \sin\theta$

18. $\cos^4\theta - \sin^4\theta = \dfrac{1}{\sec 2\theta}$

19. $\operatorname{cosec} 2\theta - \cot 2\theta = \tan\theta$

20. $\dfrac{2\tan\beta}{1+\tan^2\beta} = \sin 2\beta$

21. $\dfrac{\sin\beta + \cos\beta}{\cos\beta - \sin\beta} + \dfrac{\sin\beta - \cos\beta}{\cos\beta + \sin\beta} = 2\tan 2\beta$

22. $\dfrac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$

23. $\dfrac{\cos^3\theta - \sin^3\theta}{\cos\theta - \sin\theta} = \dfrac{2+\sin 2\theta}{2}$

24. $\operatorname{Cosec} 2\beta + \cot 2\beta = \cot\beta$

25. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

26. $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$

27. $\dfrac{\cot^2\theta - 1}{\operatorname{cosec}^2\theta} = \cos 2\theta$

28. $\dfrac{\sin 3\theta}{\cos\theta} + \dfrac{\cos 3\theta}{\sin\theta} = 2\cot 2\theta$

29. $$\frac{2}{\tan\frac{\beta}{2}+\cot\frac{\beta}{2}}=\sin\beta$$

30. $$\frac{\sin 2\alpha}{1-\cos 2\alpha}=\cot\alpha$$

31. $$\text{Sin}^4\theta=\frac{3}{8}-\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta$$

32. $$\cos\beta=\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}$$

9.6 Converting Products to Sums or Differences

Since $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ ... (i)

and $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ ... (ii)

by adding (i) and (ii)

$$2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta) \qquad (23)$$

subtracting (ii) from (i)

$$2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta) \qquad (24)$$

Now $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ (iii)

and $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ (iv)

by adding (iii) and (iv)

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta) \qquad (25)$$

subtracting (iv) from (iii)

$$2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta) \qquad (26)$$

9.7 Converting Sums or Differences to Products

From (23) we find that

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta \qquad ...(i)$$

Now, let $\alpha+\beta=\theta$ and let $\alpha-\beta=\phi$

By addition, we see that $\alpha=\frac{\theta+\varphi}{2}$, and by subtraction, we get $\beta=\frac{\theta-\varphi}{2}$

substituting in (i) we get

$$\sin\theta+\sin\varphi=2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} \qquad ...\ (27)$$

we also find from (24), that

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta \qquad ..(ii)$$

Again, let $\alpha+\beta=\theta;\ \alpha-\beta=\phi$

Then from (ii) we get

$$\sin\theta-\sin\varphi=2\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2} \qquad ...\ (28)$$

From (25), we find that

$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta \qquad .(iii)$$

As before, we let $\alpha+\beta=\theta;\ \alpha-\beta=\phi$

Then

$$\cos\theta+\cos\varphi=2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} \qquad ...\ (29)$$

Also from (26)

$$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta \qquad ...\ (iv)$$

again let $\alpha+\beta=\theta$ and $\alpha-\beta=\phi$, we get

$$\cos\theta - \cos\varphi = -2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2} \quad \ldots \quad (30)$$

summarizing,

$$\sin\theta + \sin\varphi = 2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$$

$$\sin\theta - \sin\varphi = 2\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}$$

$$\cos\theta + \cos\varphi = 2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$$

$$\cos\theta - \cos\varphi = -2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}$$

Example 1

Write the product $2\cos 95^\circ \sin 13^\circ$ as a sum or difference.

Solution

$2\cos 95^\circ \sin 13^\circ$

$= \sin(95^\circ + 13^\circ) - \sin(95^\circ - 13^\circ)$

$= \sin 108^\circ - \sin 82^\circ$

Example 2

Write the product $\cos 38^\circ \cos 61^\circ$ as a sum or difference.

Solution

$\cos 38^\circ \cos 61^\circ$

$$= \frac{1}{2}[2\cos 38^\circ \cos 61^\circ]$$

$$= \frac{1}{2}[\cos(38^\circ + 61^\circ) + \cos(38^\circ - 61^\circ)]$$

$$= \frac{1}{2}[\cos 99^\circ + \cos 23^\circ]$$

Example 3

Express $\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta$ as a product.

Solution

By grouping the first two and the last two terms.

$$(\sin 3\theta + \sin 5\theta) + (\sin 7\theta + \sin 9\theta)$$

$$= 2\sin\frac{3\theta + 5\theta}{2}\cdot\cos\frac{3\theta - 5\theta}{2} + 2\sin\frac{7\theta + 9\theta}{2}\cdot\cos\frac{7\theta - 9\theta}{2}$$

$$= 2\sin 4\theta\cos(-\theta) + 2\sin 8\theta\cos(-\theta)$$

$$= 2\sin 4\theta\cos\theta + 2\sin 8\theta\cos\theta$$

$$= 2\cos\theta[\sin 4\theta + \sin 8\theta]$$

$$= 2\cos\theta\left[2\sin\frac{4\theta + 8\theta}{2}\cos\frac{4\theta - 8\theta}{2}\right]$$

$$= 2\cos\theta[2\sin 6\theta\cos(-2\theta)]$$

$$= 4\cos\theta\sin 6\theta\cos 2\theta$$

Example 4

Show that $\dfrac{\sin\theta + \sin\varphi}{\cos\theta + \cos\varphi} = \tan\dfrac{1}{2}(\theta + \varphi)$

Solution

Taking L.H.S.

$$\frac{\sin\theta + \sin\varphi}{\cos\theta + \cos\varphi}$$

$$= \frac{2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}}{2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}}$$

$$= \frac{\sin\frac{\theta+\varphi}{2}}{\cos\frac{\theta+\varphi}{2}}$$

$$= \tan\frac{1}{2}(\theta+\varphi) = \text{R.H.S.}$$

Exercise 9.3

1. Express the following products as a sum or a difference.

(i) $2\cos 65^\circ \cos 15^\circ$ (ii) $2\sin 25^\circ \cos 35^\circ$

(iii) $2\cos 73^\circ \sin 46^\circ$ (iv) $2\sin 3A\cos A$

(v) $\cos 34^\circ \sin 28^\circ$ (vi) $\cos 30^\circ \cos 60^\circ$

(vii) $\sin 3\theta \cos 5\theta$ (viii) $\cos 7\theta \sin 5\theta$

(ix) $\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

(x) $\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

2. Convert the following sums or differences into products

(i) $\cos 56^\circ + \cos 22^\circ$ (ii) $\sin 84^\circ - \sin 76^\circ$

(iii) $\cos 84^\circ - \cos 76^\circ$ (iv) $\sin 8\theta + \sin 4\theta$

(v) $\sin\frac{5\alpha}{3} - \sin\frac{5\alpha}{6}$ (vi) $\cos\frac{3\alpha}{4} + \cos\frac{4\alpha}{3}$

(vii) $\cos\dfrac{\alpha+\beta}{2}+\cos\dfrac{\alpha-\beta}{2}$

(viii) $\sin\dfrac{\alpha+\beta}{2}-\sin\dfrac{\alpha-\beta}{2}$

Prove the following identities.

3. $\dfrac{\cos\theta-\cos\varphi}{\sin\theta+\sin\varphi}=-\tan\dfrac{1}{2}(\theta-\varphi)$

4. $\dfrac{\sin 5\theta-\sin 3\theta}{\cos 5\theta+\cos 3\theta}=\tan\theta$

5. $\sin 5\theta+2\sin 3\theta+\sin\theta=4\sin 3\theta\cos^2\theta$

6. $\dfrac{\sin 5\alpha+\sin 3\alpha}{\cos 5\alpha-\cos 3\alpha}=-\cot\alpha$

7. $\dfrac{\cos\beta+\cos 9\beta}{\sin\beta+\sin 9\beta}=\cot 5\beta$

8. $\dfrac{\sin 4A-\sin 2A}{\cos 4A+\cos 2A}=\tan A$

9. $\dfrac{\sin 3\alpha-\sin 5\alpha}{\cos 5\alpha-\cos 3\alpha}=\cot 4\alpha$

10. $\dfrac{\sin 3\varphi-\sin\varphi}{\cos^2\varphi-\sin^2\varphi}=2\sin\varphi$

11. $\dfrac{\cos\alpha-\cos\beta}{\cos\alpha+\cos\beta}=-\dfrac{\tan\frac{1}{2}(\alpha+\beta)}{\cot\frac{1}{2}(\alpha-\beta)}$

12. $\dfrac{\sin\theta+\sin 2\theta+\sin 3\theta}{\cos\theta+\cos 2\theta+\cos 3\theta}=\tan 2\theta$

13. $\sin 4\theta-\sin 8\theta+\sin 12\theta=4\cos 6\theta\sin 2\theta\cos 4\theta$

Show that

14. $\dfrac{\sin 75^o - \sin 15^o}{\cos 75^o + \cos 15^o} = \dfrac{1}{\sqrt{3}}$

15. $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \dfrac{1}{16}$

16. $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \dfrac{1}{16}$

17. $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \dfrac{3}{16}$

Chapter 10

Trigonometric Functions

10.1 Domain and range of Trigonometric Functions.

We know as to what is meant by domain and range of any function. We now find the domains and ranges of the six trigonometric functions.

10.2 Domains and Ranges of the Sine and Cosine

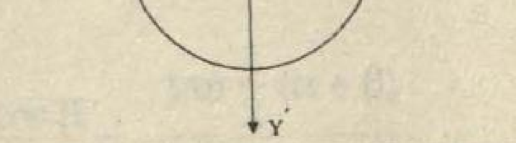


Fig. 10.1

If a ray $\overrightarrow{OP}$ starting from the initial position $\overrightarrow{OX}$ revolves continuously in the counter clockwise direction, it generates angles of all positive measures. This means that corresponding to every non-negative real number θ, there is one and only one angle XOP. Similarly when the ray OP revolves in the clockwise direction, it generates angles of all negative measures. Therefore, corresponding to every negative real number θ, there is one and only one angle ∠ XOP. As a consequence, corresponding to every real number θ, there is one and only one ∠ XOP.

For convenience $m\,\overline{OP}$ may be taken as unity. When the ray OP revolves about O in anticlockwise or clockwise direction, the point P always lie on a unit circle with centre at O.

If in any position of $\overrightarrow{OP}$, m ∠ XOP is θ and the coordinates of P are (x, y), then

$$\sin\theta = \frac{y}{1} = y,$$

$$\text{and} \qquad \cos\theta = \frac{x}{1} = x$$

From the above discussion it is clear that corresponding to any real number θ, there is one and only one position of the point P and, therefore, one and only one value for each of the coordinates x, y i.e., one and only one value for each of sin θ and cos θ.

Hence sin θ and cos θ are functions, the *domain* of each of which is R, the set of real numbers.

To find the range of either function, we recall the identity

$$\sin^2\theta + \cos^2\theta = 1$$

from which it follows that each of $\sin\theta$ and $\cos\theta$ is numerically less than or equal to 1, that is:

$$-1 \le \sin\theta \le 1$$

and $$-1 \le \cos\theta \le 1$$

The same fact is clear from fig. (1).

As the revolving ray $\overrightarrow{OP}$ starts from the initial position $\overrightarrow{OX}$ and revolves upto the position $\overrightarrow{OY}$, y or $\sin\theta$ increases continuously from 0 to 1. When $\overrightarrow{OP}$ revolves from the position $\overrightarrow{OY}$ to $\overrightarrow{OX'}$, y or $\sin\theta$ decreases from 1 to 0.

Similarly in the third quadrant y decreases from 0 to -1 and in the fourth quadrant y increases from -1 to 0.

Thus y or $\sin\theta$ always lies between -1 and 1, including both these values.

Similarly it can be seen that the values of x or $\cos\theta$ also lie in the same interval. Thus the range of the sine as well as cosine function is $\{x \mid -1 \le x \le 1\}$.

10.3. Domains and Ranges of Tangent and Cotangent Functions

Referring to fig. (1):

Corresponding to any value of θ, $\tan\theta$ has a finite and definite value except when $x = 0$, i.e., when $\overrightarrow{OP}$ coincides with $\overrightarrow{OY}$ or $\overrightarrow{OY'}$.

In such a case

$$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Therefore, $\tan\theta$ has a unique value except when θ is an odd multiple of $\frac{\pi}{2}$, i.e., it is of the form $(2k+1)\frac{\pi}{2}$. (where k is any integer)

Hence the domain of $\tan\theta = R - \left\{x \mid x = (2k+1)\frac{\pi}{2}, k \in Z\right\}$

Now $\quad \cot\theta = \frac{x}{y}$

$\therefore$ $\cot\theta$ has finite and definite value except when $y = 0$. This happens when $\overrightarrow{OP}$ coincides with $\overrightarrow{OX}$ or $\overrightarrow{OX'}$ i.e. when

$\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\therefore$ domain of $\cot\theta = R - \{x \mid x = k\pi, k \in Z\}$.

We now consider the ranges of $\tan\theta$ and $\cot\theta$.

As already stated $\tan\theta$ is not defined at $\theta = \frac{\pi}{2}$. It is, however, clear that as θ approaches $\frac{\pi}{2}$ through values less than $\frac{\pi}{2}$, x approaches 0 and therefore $\tan\theta = \frac{y}{x}$ becomes larger and larger. This state of affairs is expressed by saying that

$\tan\theta \to \infty$ (tan θ tends to infinity)

as $\theta \to \frac{\pi}{2} - 0$ (0 here stands for a very very small number)

As soon as $\overrightarrow{OP}$ crosses $\overrightarrow{OY}$, x becomes negative but remains very small. Since y is +ve in the second quadrant,

$$\tan\theta = \frac{y}{x}$$

becomes very large in magnitude but –ve in sign. This situation is expressed by saying that:

$$\tan\theta \to -\infty$$

$$\text{as} \qquad \theta \to \frac{\pi}{2} + 0$$

Clearly, tan θ decreases numerically in the second quadrant and becomes 0 when $\theta = \pi$.

In the third quadrant, as θ varies from π to $\frac{3\pi}{2} - 0$, tan θ varies from 0 to ∞. (tan θ is +ve in the 3rd quadrant).

As $\overrightarrow{OP}$ crosses $\overrightarrow{OY'}$, x becomes positive but y remains –ve and very small. Therefore, in the fourth quadrant tan θ varies from $-\infty$ to 0.

In this way tan θ assumes all real values. Hence Range of tan θ = R, the set of real numbers.

Similarly it can be seen that although Cot θ is not defined when $\theta = k\pi$, it attains all real values. So the range of Cot θ is also R.

10.4. Domains and Ranges of Secant and Cosecant

With reference to Fig. (10.1)

$$\sec\theta = \frac{1}{x}$$

∴ sec θ is defined for all real values of x except when $x = 0$. This will be so when $\overrightarrow{OP}$ coincides with $\overrightarrow{OY}$ or $\overrightarrow{OY'}$ and then

θ will be an odd multiple of $\frac{\pi}{2}$.

$$\therefore \text{Domain of rec } \theta = R - \left\{x \mid x = (2k+1)\frac{\pi}{2}, k \in Z\right\}$$

Also $\operatorname{cosec}\theta = \frac{1}{y}$

$\therefore$ cosec θ will be undefined when $y = 0$ i.e. when $\overrightarrow{OP}$ coincides with $\overrightarrow{OX}$ or $\overrightarrow{OX'}$.

This will be so when θ is an integral multiple of π.

$$\therefore \text{Domain of cosec } \theta = R - \{x \mid x = k\pi, k \in Z\}$$

Again consider $\sec\theta = \frac{1}{x}$

In the first quadrant θ increases from 0 to $\frac{\pi}{2} - 0$ and x varies from 1 to $0 + 0$. Therefore, sec θ varies from 1 to ∞.

In the second quadrant x varies from $0 - 0$ to 1, so $\sec\theta = \frac{1}{x}$ varies from $-\infty$ to -1. Similarly it can be seen that in the third quadrant sec θ varies from -1 to $-\infty$ and in the fourth quadrant from ∞ to 1.

Thus sec θ attains all real values except those between -1 and 1 (excluding both these values).

$$\therefore \text{range of } \sec\theta = R - \{x \mid -1 < x < 1\}.$$

By considering the variations of $\operatorname{cosec}\theta = \frac{1}{y}$, it can be easily seen that the range of cosec θ is the same as that of sec θ.

The results derived above are compactly shown below in the form of a table.

FUNCTION	DOMAIN	RANGE
sine	R = Set of all real numbers	$\{x \mid x \in R \wedge -1 \le x \le 1\}$
cosine	R	$\{x \mid x \in R \wedge -1 \le x \le 1\}$
tangent	$R - \{x \mid x = (2k+1)\pi/2,\ k \in Z\}$	R
cotangent	$R - \{x \mid x = k\pi,\ k \in Z\}$	R
secant	$R - \{x \mid x = (2k+1)\pi/2,\ k \in Z\}$	$R - \{x \mid -1 < x < 1\}$
cosecant	$R - \{x \mid x = k\pi,\ k \in Z\}$	$R - \{x \mid -1 < x < 1\}$

10.5. Periods of Trigonometric Functions

Definition

A function f is said to be *periodic* if for every x belonging to its domain D there exists a positive number p such that

$$x + p \in D$$

$$\text{and} \qquad f(p + x) = f(x).$$

If p is the least positive number satisfying these conditions, then it is called the period of f.

Good examples of periodic functions are provided by the trigonometric functions.

10.5.1. Periods of sine and cosine

Let us find the period of the sine function.

Suppose p is the required period.

Then $\sin(\theta + p) = \sin\theta$, for all real values of θ

(Recall that Domain $\sin\theta = R$)

In particular, the last equation should hold for $\theta = 0$

$(\because 0 \in R)$

$\therefore \sin p = \sin 0 = 0.$

But $\sin p = 0 \Rightarrow p = 0, \pi, 2\pi, 3\pi, \ldots$

(We consider positive values of p only)

Let us check if π is the required period.

$$\sin(\theta + \pi) = -\sin\theta \neq \sin\theta.$$

$\therefore$ π is not the required period.

Next we check 2π:

$$\sin(\theta + 2\pi) = \sin\theta.$$

$\therefore$ 2π is the desired period of the sine function.

Let us now find the period of the cosine function. If p is the required period,

$$\cos(p + \theta) = \cos\theta \text{ for all } \theta \in R$$

Let $\theta = \frac{\pi}{2}$ $\quad$ ($\pi/2 \in R$), then

$$\cos\left(p + \frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$\therefore \quad \cos\left(p + \frac{\pi}{2}\right) = 0$$

$$\Rightarrow -\sin p = 0 \Rightarrow p = 0, \pi, 2\pi, 3\pi, \ldots$$

Now $\cos(\pi + \theta) = -\cos\theta$.

$\therefore$ π is not the required period.

$\because$ $\cos(2\pi + \theta) = \cos\theta$,

$\therefore$ 2π is the required period.

10.5.2. Periods of Tangent and Cotangent

Using methods similar to those employed for finding the periods of the sine and cosine functions respectively it can be proved that the period of the tangent as well as cotangent function is π.

10.5.3. Periods of Secant and Cosecant

Let p be the period of the secant.

Then $\sec(p + \theta) = \sec\theta$ for all $\theta \in$ dom. $\sec\theta$

$(0 \in \text{dom. } \sec\theta)$

Let $\theta = 0$

So that $\quad \sec p = \sec 0 = 1$

$$\Rightarrow \text{p} = 0, 2\,\pi, 4\,\pi, 6\,\pi, \ldots$$

Clearly 2π is the required period.

$[\sec(2\pi + \theta) = \sec\theta]$

Finally, suppose p is the period of the cosecant, so that

$\text{cosec}(p + \theta) = \text{cosec}\,\theta$ for all $\theta \in$ dom. cosec θ.

Let $\theta = \frac{\pi}{2}$, then

$$\text{cosec}\left(p + \frac{\pi}{2}\right) = \text{cosec}\,\frac{\pi}{2} = 1$$

$$\Rightarrow \sec p = 1.$$

$$\Rightarrow \text{p} = 0, 2\,\pi, 4\,\pi, 6\,\pi, \ldots$$

$\therefore$ 2π is the period of cosec as well.

Example 1

Find those real numbers which are not included in the domain of

(i) $\tan 2x$, (ii) $\text{cosec}\,\frac{1}{3}x$.

Solution

(i) Real numbers excluded from the domain of tan $2x$ are given by

$$2x = (2k+1)\frac{\pi}{2}, \qquad \text{where } k \in Z.$$

$$\therefore x = \left(k + \frac{1}{2}\right)\frac{\pi}{2}, \qquad \text{where } k \in Z.$$

(ii) Real numbers excluded from the domain of cosec $\frac{1}{3}x$ are given by

$$\frac{1}{3}x = k\pi, \qquad \text{k} \in Z$$

$$\therefore x = 3k\pi, \qquad k \in Z.$$

Example 2

Find the range of (i) sin $2x$ (ii) 2 cos x.

Solution

(i) The equation $\sin^2\theta + \cos^2\theta = 1$

is true for all real values of θ. Replacing θ by $2x$,

$$\sin^2 2x + \cos^2 2x = 1.$$

is also true for all real values of x.

$$\therefore -1 \le \sin 2x \le 1, \quad -1 \le \cos 2x \le 1.$$

$\therefore$ range of sin $2x$ is also the set

$$\{x \mid -1 \le x \le 1, x \in R\}$$

(ii) $\quad -1 \le \cos x \le 1$

$$\therefore -2 \le 2\cos x \le 2$$

So range of $2\cos x = \{x \mid -2 \leq x \leq 2\}$

i.e. the closed interval $[-2, 2]$.

Example 3

What is the period of (i) $\sin 3x$, (ii) $\tan \frac{1}{5}x$?

Solution

(i) $$\sin 3x = \sin(3x + 2\pi)$$
$$= \sin 3\left(x + \frac{2\pi}{3}\right)$$

Value of $\sin 3x$ repeats when x is increased by $\frac{2\pi}{3}$

$\therefore$ required period $= \frac{2\pi}{3}$

(ii) $$\tan \frac{1}{5}x = \tan\left(\frac{1}{5}x + \pi\right)$$
$$= \tan \frac{1}{5}(x + 5\pi)$$

Value of $\tan \frac{x}{5}$ repeats after an interval of 5π

$\therefore$ required period $= 5\pi$.

Exercise 10.1

State the domain and period of each of the following functions:-

1. $\sin x$ 2. $\tan x$ 3. $\operatorname{cosec} x$

4. $6\sin x$ 5. $\cos 6x$ 6. $\tan \frac{4}{3}x$

7. $5 \cot x$ 8. $\text{cosec}\,\frac{x}{4}$ 9. $4 \cos 4x$

State the range of each function:-

10. $\sin 4x$ 11. $5 \cos 5x$ 12. $3 \tan 3x$

13. $\sin \pi x$ 14. $\cos \frac{\pi}{2} x$ 15. $\tan \frac{\pi}{4} x$

16. $\cot \frac{\pi}{3} x$ 17. $\sec (2\pi x + 5)$ 18. $\text{cosec}\left(\frac{\pi}{5} x + \right.$

10.6. Graphs of Trigonometric Functions

In order to graph a function $y = f(x)$, we give a number of permissible values to x (values belonging to its domain and the specified interval) and obtain the corresponding values of y. The ordered pairs of the values of (x, y) so obtained are plotted as points on the coordinate plane. Joining these points by a curve we get the required graph.

10.6.1. The Sine Graph

In place of denoting the angular measure by θ or α, we denote it by x because a distance along or parallel to the x-axis is usually denoted by x.

Let us draw the graph of:

$$y = \sin x, \text{ where } 0^o \leq x \leq 360^o,$$

$$\text{or} \quad 0 \leq x \leq 2\pi$$

Values of x are represented along x-axis on a convenient scale which is often obvious from the graph and the corresponding values of y are represented parallel to the y-axis.

We know that the domain of the sine function is R, the set of all real numbers. Therefore, all values of x in the given interval are permissible values. Starting with $x = 0$, we give values to x at each interval of 30° ($\pi/6$), and find the corresponding values of y or $\sin x$.

Values of $\sin x$ in the second, third and fourth quadrants are found by using the formulae.

$$\sin(180^\circ - x) = \sin x,\ \sin(180^\circ + x) = -\sin x,$$

$$\sin(360^\circ - x) = \sin(-x) = -\sin x.$$

Thus, for example

$$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^0 = \frac{\sqrt{3}}{2} = .87$$

$$\sin 210^\circ = \sin(180^\circ + 30^\circ) = -\sin 30^\circ = -\frac{1}{2} = -.5$$

$$\sin 300^\circ = \sin(360^\circ - 60^\circ) = \sin(-60) = -\sin 60^\circ = -.87$$

Using the above formulae, we get the following table:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$y = \sin x$	0	.5	.87	1	.87	.5	0	−.5	−.87	−1	−.87	−.5	0

Plotting the corresponding values of x and y on the coordinate plane we get the graph shown in Fig. 10.2.

Remarks

1 The whole graph lies between the lines $y = -1$ and $y = 1$. This is in conformity with the fact that the range of sine is the closed interval $[-1, 1]$ or the set $\{y \mid -1 \leq y \leq 1\}$.

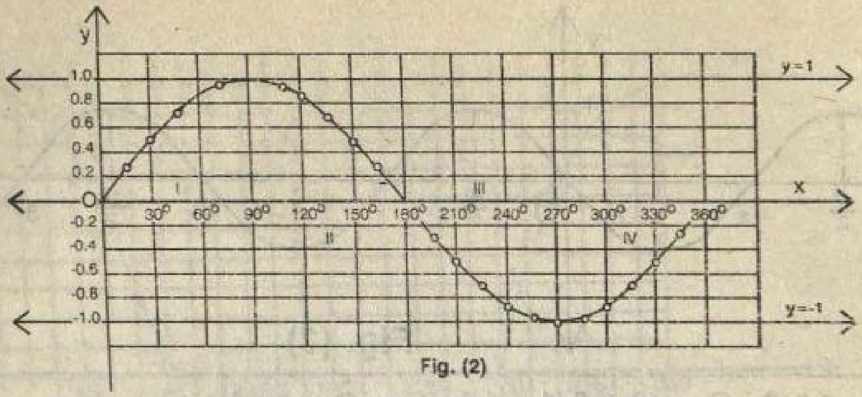


Fig. (2)

Fig. 10.2 Graph of sin x from 0 to 2 π.

(2) sin x is positive in the first two quadrants and negative in the remaining two quadrants.

(3) The graph crosses the x-axis at the points $x = 0, \pi$ and 2π. (At these points the value of sin x is zero).

(4) The vertical distance (distance parallel to the y-axis) between the points of maximum and minimum values of the sine function is 2 units. Half of this distance is called *Amplitude* of the sine function. Amplitude of $y = \sin x$ is, therefore, 1.

(5) Since the period of the sine function is 2π, its graph from -2π to 0 or from 2π to 4π will be of the same shape as from 0 to 2π. This is borne out by the graph in Fig. 10.3.

In case of a periodic function it is sufficient to draw the graph in an interval equal to one period only. Then the graph can be easily extended to any other interval.

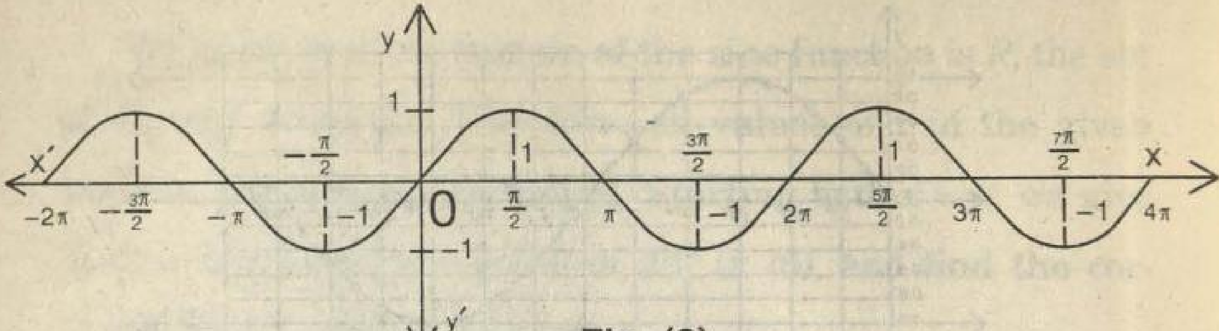


Fig. (3)

Fig. 10.3 *Graph of sin x from − 2 π to 4 π*

In the above sine graph, portions of the graph for the intervals -2π to 0, 0 to 2π and 2π to 4π are exactly similar.

(6) *Variations* of sin x in the four quadrants are also evident from the graph. They are stated alongwith the variations of the remaining functions in the next section.

10.6.2. The Cosine Graph (0 to 360°)

The following table can be easily formed for the values of x and cos x:

x	0	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos x	1	.87	.5	0	−.5	−.87	−1	−.87	−.5	0	.5	.87	1

Plotting the points corresponding to the values we get the graph as in Fig 10.4.

As in the case of the sine graph, the cosine graph can also be easily extended to the intervals -2π to 0 and 2π to 4π. This can be seen from the graph as in Fig 10.5.

10.6.3. The Tangent Graph

We know that the tangent function is not defined at the

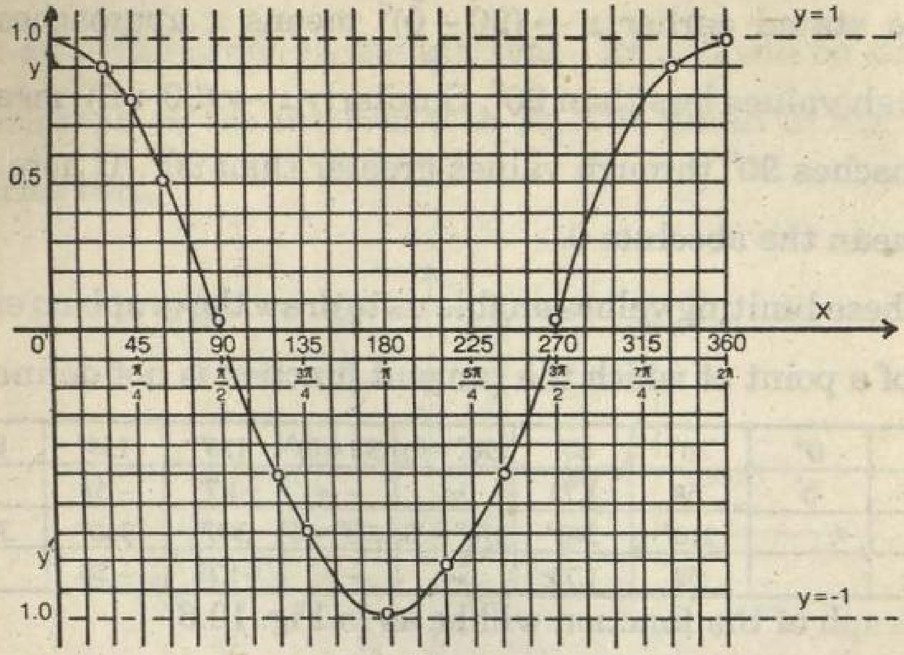


Fig. (4)

Fig. 10.4

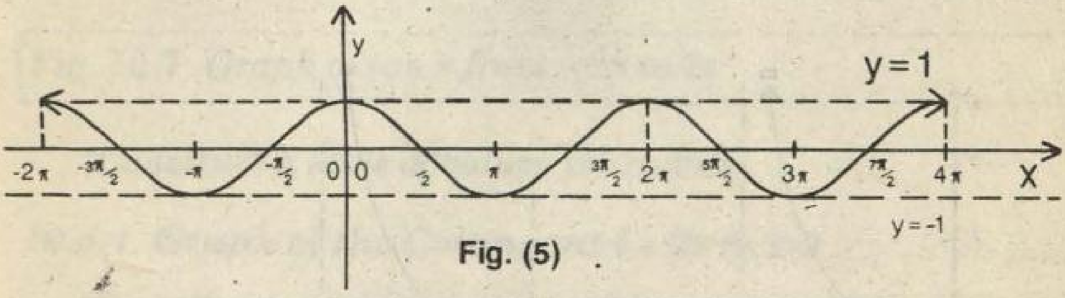


Fig. (5)

Fig. 10.5 *Graph of cosine x from* -2π to 4π

points $x = 90^\circ, 270^\circ$ etc. However as already explained:-

$$\text{as } x \to (90^o - 0^o),\ \tan x \to \infty \text{ (First quadrant)}$$

$$\text{as } x \to (90^o + 0^o),\ \tan x \to -\infty \text{ (Second quadrant)}$$

$$\text{as } x \to (270^o - 0^o),\ \tan x \to \infty \text{ (Third quadrant)}$$

$$\text{as } x \to (270^o + 0^o),\ \tan x \to -\infty \text{ (Fourth quadrant).}$$

As stated earlier $x \to (90-0)^o$ means x approaches 90^o through values less than 90^o. Similarly $x \to (90+0)^o$ means x approaches 90^o through values greater than 90^o. '0' here does not mean the absolute 0.

These limiting values enable us to draw the graph on either side of a point at which the tangent function is not defined.

x	0^o	30^o	60^o	(90^o-0)	$(90+0)^o$	120^o	150^o	180^o
$\tan x$	0	.58	1.73	∞	$-\infty$	-1.73	$-.58$	0
x		210^o	240^o	270^o-0	270^o+0	300^o	330^o	360^o
$\tan x$		.58	1.73	∞	$-\infty$	-1.73	$-.58$	0

Graph of the function will be as in Fig. 10.6.

We know that the period of the tangent function is π. The graph confirms the fact. The portions of the graph in the intervals o to π and π to 2π are quite similar.

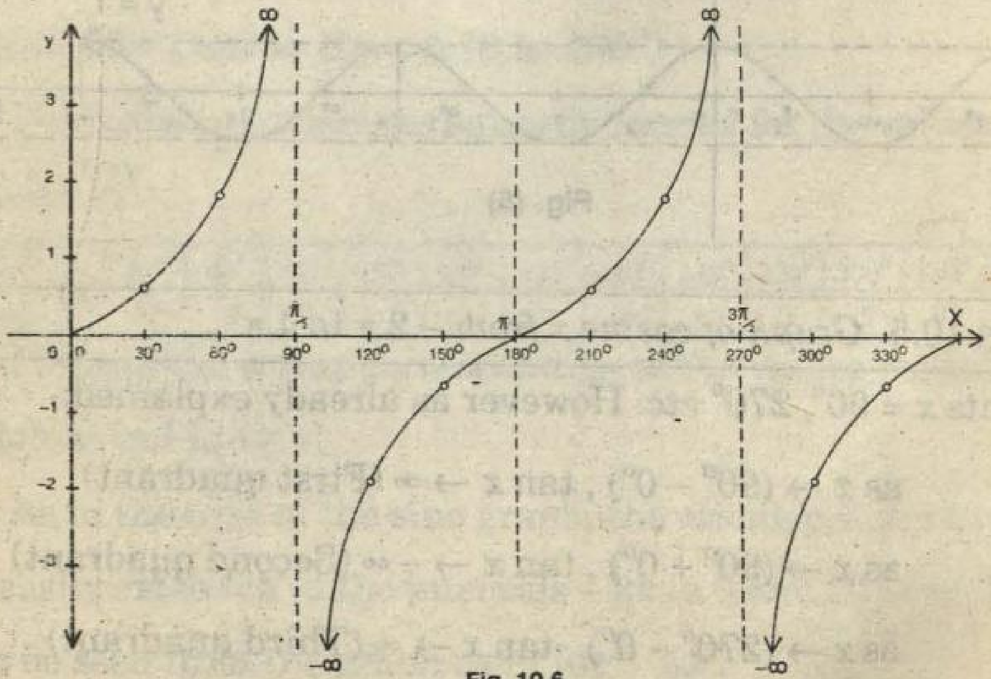


Fig. 10.6

Fig. 10.6

We can easily extend the graph in the negative direction of the x-axis. The graph in the interval -2π to 0 will be similar to the graph in the interval 0 to 2π. The graph in Fig. 10.7 confirms this.

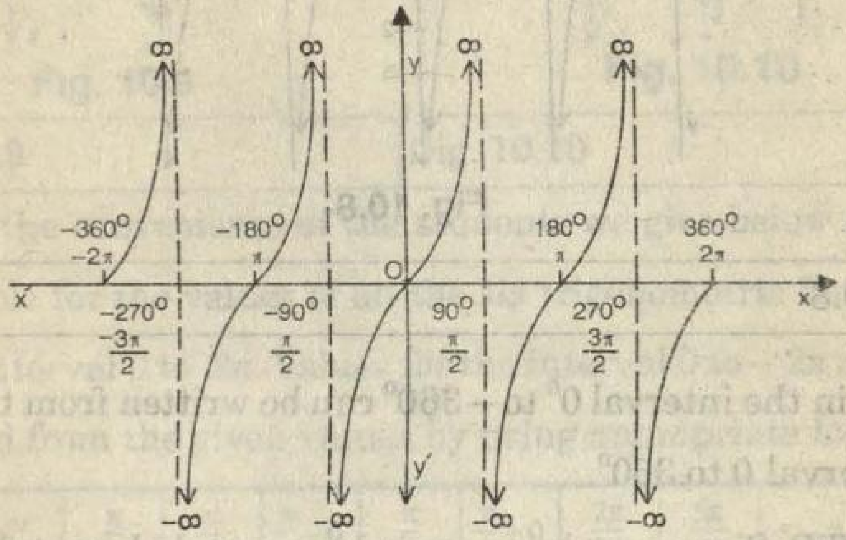


Fig. 10.7

Fig. 10.7 *Graph of tan x from -2π to 2π*

(Construct a table of values yourselves).

10.6.4. Graph of the Cotangent (-2π to 2π)

The following table can be easily constructed for values of x and $\cot x$.

x	(0 + 0)°	30°	60°	90°	120°	150°	(180 – 0)°
$\cot x$	∞	1.73	.58	0	–.58	–1.73	– ∞
x	(0 – 0)°	–30°	–60°	–90°	–120°	–150°	(–180+0)°
$\cot x$	– ∞	–1.73	–.58	0	.58	1.73	∞
x	(180 + 0)°	210°	240°	270°	300°	330°	360°
$\cot x$	∞	1.73	.58	0	–.58	–1.73	– ∞
x	–180° – 0°	–210°	–240°	–270°	–300°	–330°	–360°
$\cot x$	– ∞	–1.73	–.58	0	.58	1.73	∞

Recall that $\cot(-\theta) = -\cot\theta$. By using this formula values

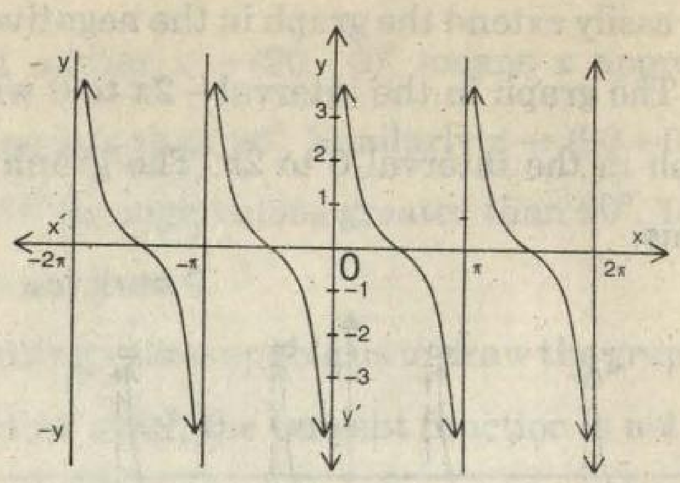


Fig. 10.8

Fig. 10.8

of cot x in the interval 0° to -360° can be written from those in the interval 0 to 360°.

NOTE: Since cot x has a period π, we could have tabulated its values in the interval 0 to π only and the graph in this interval could have been repeated in the intervals -2π to $-\pi$, $-\pi$ to 0, π to 2π.

11.6.5. Graph of the Secant

Graph of the secant function in the interval 0 to 2π is shown below (10;. 9). The students should prepare the table of values themselves. On account of the periodicity of the function, graph in the interval -2π to 0 will be similar to that given for the interval 0 to 2π.

10.6.6. Graph of the Cosecant

Required graph is shown in Fig. (10. 10). The students should prepare the requisite table of values themselves.

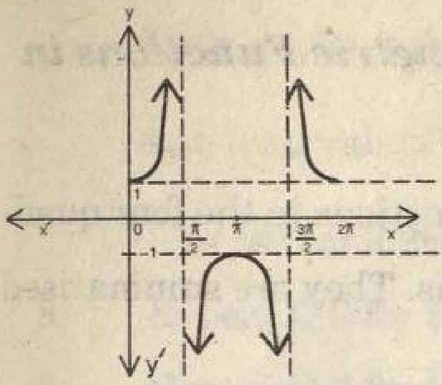


Fig. 10.9

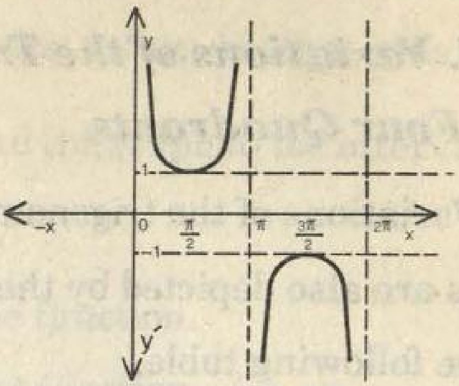


Fig. 10.10

Fig. 10.9	Fig. 10.10

For the convenience of the students we give below a compact table for the values of all the six trigonometric functions in the interval 0 to 2π. Values for the interval 0 to – 2π may be obtained from the given values by using appropriate formula.

x	0^0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0$	$\frac{\pi}{2}$	$\frac{\pi}{2}+0$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	π
sin x	0	.5	.87		1		.87	.5		0
cos x	1	.87	.5		0		–.5	–.87		–1
tan x	0	.58	1.73	∞	×	– ∞	–1.73	–.58		0
cot x	∞ at (0+0)	1.73	.58		0		–.58	–1.73	– ∞	×
sec x	1	1.15	2	∞	×	– ∞	–2	–1.15		–1
cosec x	∞ at (0+0)	2	1.15		1		1.15	2	∞	×

x	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0$	$\frac{3\pi}{2}$	$\frac{3\pi}{2}+0$	$\frac{5\pi}{3}$	$\frac{5\pi}{6}$	2π
sin x		–.5	–.87		– 1		– .87	–.5	0
cos x		–.87	–.5		0		.5	.87	1
tan x		.58	1.73	∞	×	– ∞	–1.73	–.58	0
cot x	∞	1.73	.58		0		–.58	–1.73	–∞ at $2\pi-0$
sec x		–1.15	–2	– ∞	×	∞	2	1.15	1
cosec x	– ∞	–2	–1.15		–1		–1.15	–2	–∞ at $2\pi-0$

10.7. Variations of the Trigonometric Functions in the Four Quadrants.

Variations of the trigonometric functions in the four quadrants are also depicted by their graphs. They are summarised in the following table.

I = Increase		D = Decrease		
Quad.	1st	2nd	3rd	4th
sin x	I: 0 to 1	D: 1 to 0	D: 0 to – 1	I: – 1 to 0
cos x	D: 1 to 0	D: 0 to – 1	I: – 1 to 0	I: 0 to 1
tan x	I: 0 to ∞	I: – ∞ to 0	I: 0 to ∞	I: – ∞ to 0
cot x	D: ∞ to 0	D: 0 to – ∞	D: ∞ to 0	D: 0 to – ∞
sec x	I: 1 to ∞	I: – ∞ to – 1	D: –1 to – ∞	D: ∞ to 1
cosec x	D: ∞ to 1	I: 1 to ∞	I: – ∞ to – 1	D: –1 to – ∞

Exercise 10.2

1. Draw the graph of the sine function in the interval 0 to 2π by tabulating its values at the ends of sub-intervals of $\frac{\pi}{6}$.
2. Repeat Q.1 for the cosine function.
3. Repeat Q.1 for the secant function.
4. Repeat Q.1 for the cosecant function.
5. Draw the graph of the tangent function in the interval 0 to π. Extend the graph to the intervals π to 2π, -2π to $-\pi$ and $-\pi$ to 0.
6. Repeat Q.5 for the cotangent function.
7. Draw the graph of the sine function in the finterval

0 to 2π by tabulating its values at the ends of sub-intervals of $\frac{\pi}{2}$. Extend the graph to the intervals -2π to 0 and 2π to 4π.

8. Repeat Q.7 for the cosine function.
9. Repeat Q.7 for the secant function.
10. Repeat Q.7 for the cosecant function.
11. Describe the variations of the values of sine and cosecant functions in the four quadrants.
12. Repeat Q.11 for the tangent and cotangent functions.
13. Repeat Q.11 for the cosine and secant functions.

Chapter 11

Solution of triangles and Area of a Triangle

For a given triangle ABC, we shall use *a*, *b*, *c* to denote the measure of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively and α, β, γ to denote the measures of angles A, B, C respectively.

11.1. Solution Of Right Triangles

If any two of α, β, *a*, *b* and *c* are known then the remaining can be found by the use of trigonometric functions and their tables. This process is called solving the triangle. In the following we shall use α and β for the acute angles and γ for right angle of right-angled triangle.

Example 1

Solve the right triangle in which

$$\alpha = 34^o\ 17',\ b = 31.75$$

Solution

We have to find *a*, *c* and β.

$$\text{Now } \beta = 90^o - 34^o\ 17' = 55^o\ 43'$$

To find a, we use $\tan \alpha$.

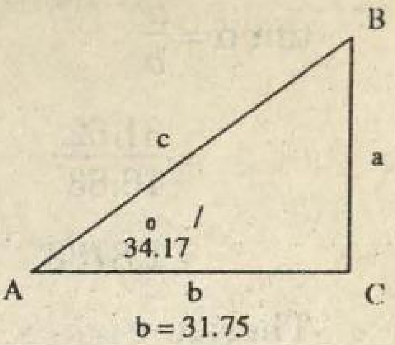


Fig. 11.1

$$\tan \alpha = \frac{a}{b}$$

$$\text{i.e. } \frac{a}{b} = \tan 34^o\ 17'$$

Therefore

$$a = 31.75 \tan 34^o\ 17'$$

$$= 31.75\,(.6817)$$

$$= 21.6439$$

$$= 21.64$$

To find c, we use $\cos \alpha$.

$$\cos \alpha = \frac{b}{c}$$

$$\text{i.e. } \cos 34^o\ 17' = \frac{31.75}{c}$$

Therefore $$c = \frac{31.75}{\cos 34^o\ 17'}$$

$$= \frac{31.75}{.8219}$$

$$= 38.63$$

Example 2

Solve the right triangle in which

$$a = 31.62,\ b = 16.83$$

Solution

We have to find α, β and c.

To find α, we use $\tan \alpha$.

$$\tan\alpha = \frac{a}{b}$$

$$= \frac{31.62}{16.83}$$

$$= 1.8788$$

Therefore

$$\alpha = 61^\circ\, 58'\, 32''$$

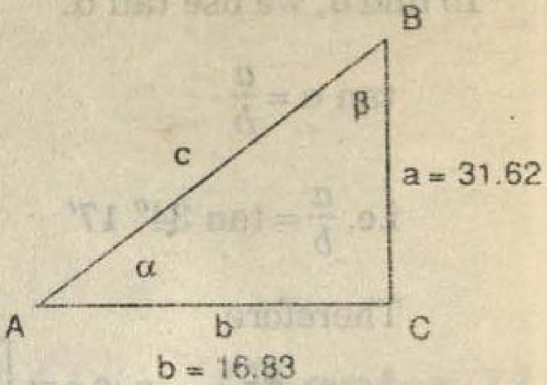


Fig. 11.2

so

$$\beta = 90^o - 61^o\, 58'\, 32''$$

$$= 28^o\, 1'\, 28''$$

To find c, we use sin α

$$\sin\alpha = \frac{a}{c}$$

$$\text{i.e. } \sin 61^o\, 58'\, 32'' = \frac{31.62}{c}$$

$$\text{Therefore } c = \frac{31.62}{\sin 61^o\, 58'\, 32''}$$

$$= \frac{31.62}{.8827}$$

$$= 27.91$$

Exercise 11.1

Solve the right triangles ABC in which $\gamma = 90^o$.

1. $a = 250, \alpha = 42^o\, 25'$
2. $a = 432, \alpha = 35^o\, 36'$
3. $b = 234, \beta = 40^o\, 45'$

4. $b = 536, \beta = 55^\circ 40'$
5. $c = 232, \alpha = 52^\circ 46'$
6. $c = 410, \beta = 46^\circ 32'$
7. $a = 450, b = 340$
8. $c = 540, a = 380$
9. $c = 632, b = 240$
10. $a = 4.25, b = 5.35$
11. $b = 312, \alpha = 23^\circ 42'$
12. $a = 212, \beta = 40^\circ 55'$

11.2. Heights and Distances Involving Right Triangles

Sometimes we deal with problems in which we have to find heights and distances. The solution of these problems are generally the same as that of solving the right triangles. Before dealing with such problems we shall define first:

(i) Angle of Elevation

The angle AOP which the ray from an observer's eye at *O* to an object at *P* at a higher level makes with horizontal ray *OA* through an observers eye at *O* is called the angle of elevation.

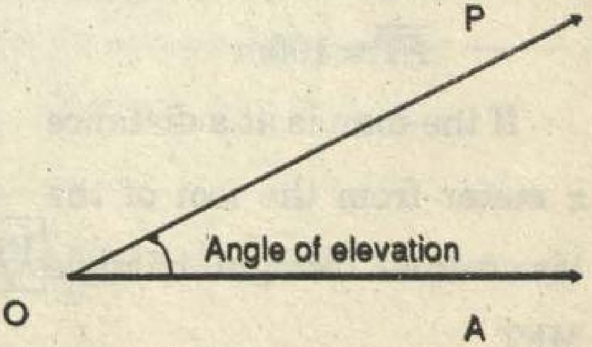


Fig. 11.3

(ii) Angle of Depression

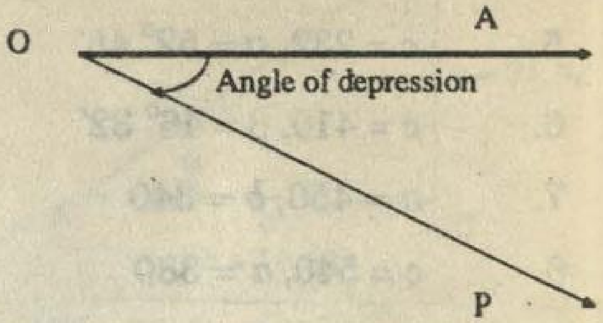


Fig. 11.4

The angle AOP which the ray from an observer's eye at O to an object at P at a lower level makes with the horizontal ray OA through an observer's eye at O is called the angle of depression. The following examples will illustrate, how to find a distance as height in a given problem.

Example 1

Find the distance of a man from the foot of a tower 100 m high if the angle of elevation of its top as observed by the man is $52^\circ 30'$

Solution

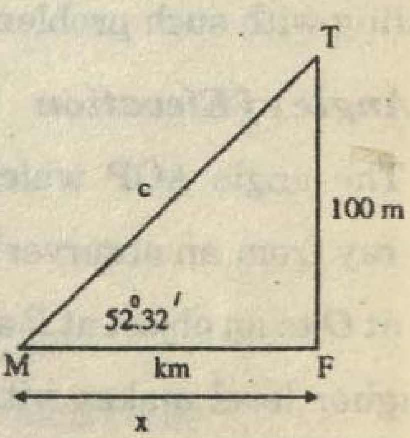


Fig. 11.5

Let M be the position of the man and F be the foot of the tower. Height of the tower is

$$\overline{FT} = 100m$$

If the man is at a distance x meter from the foot of the tower then in right triangle MFT

$$\tan 52^\circ\, 30' = \frac{\overline{FT}}{\overline{MF}}$$

$$\text{or } \overline{MF} = \frac{\overline{FT}}{\tan 52^\circ\, 30'}$$

$$\text{or } x = \frac{100}{1.3092}$$

$$= 76.73 \text{ m}$$

Example 2

From the two successive positions on a straight road 1000 meter apart a man observes that the angle of elevation of the top of a tower directly ahead of him is of 12° 10′ and 42° 35′. How high is the tower above the road.

Solution

Let A and B be the two successive positions of a man on the road such that

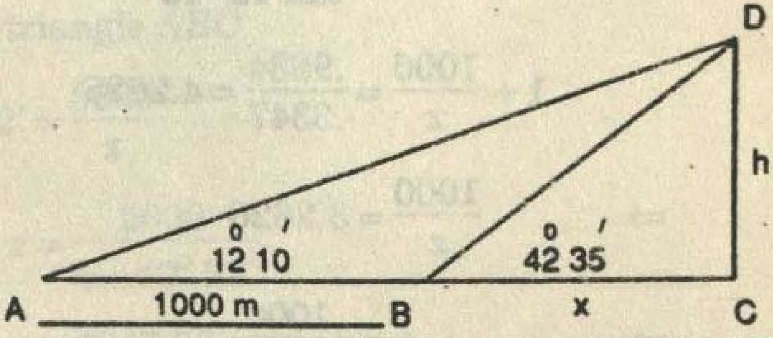


Fig. 11.6

$$m\,\overline{AB} = 1000m$$

Let C and D be the foot and top of the tower respectively and h the height of the tower.

In right triangle ACD

$$\tan 12^\circ\, 10' = \frac{\overline{CD}}{\overline{AC}}$$

$$= \frac{h}{\overline{AB} + \overline{BC}}$$

$$= \frac{h}{1000 + x}$$

$$\text{or } \frac{h}{x + 1000} = \tan 12^o\ 10' \quad \ldots \quad \text{(i)}$$

In right triangle BCD

$$\tan 42^o\ 35' = \frac{\overline{CD}}{\overline{BC}}$$

$$\text{or} \quad \frac{h}{x} = \tan 42^o\ 35'.. \quad \text{(ii)}$$

Dividing (ii) by (i)

$$\frac{x + 1000}{x} = \frac{\tan 42^o\ 35'}{\tan 12^o\ 10'}$$

$$1 + \frac{1000}{x} = \frac{.9834}{.3347} = 4.2626$$

$$\Rightarrow \quad \frac{1000}{x} = 3.2626$$

$$\Rightarrow \quad x = \frac{1000}{3.2626}$$

$$= 306.50\ m$$

$$\text{From (i) } \frac{h}{306.50} = \tan 42^o\ 35'$$

$$\text{or} \quad h = (306.50)\ (.9834)$$

$$= 301.41 \text{ m}$$

Example 3

The measure of the angle of depression of an airport as observed by a pilot while flying at a height of 5000 meters is $40^o\,32'$. How far is the plane from a point directly over the airport.

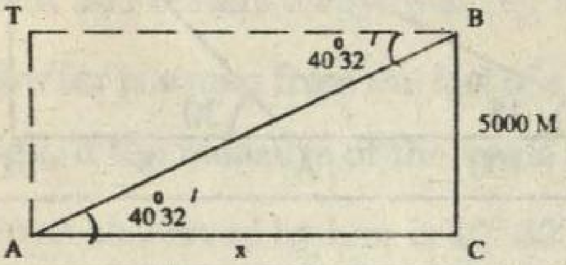


Fig. 11.7

Solution

The pilot is at the point B.

$$m\overline{CB} = 5000 \text{ meters}$$

from the right triangle ABC

$$\tan 40^o\,32' = \frac{5000}{x}$$

$$x = \frac{5000}{\tan 40^o\,32'}$$

$$= 5847.36$$

Hence the distance of the pilot from the point T directly over the airport at A is 5847.36 meters.

Example 4

The measure of the angle of elevation of the top of a tower is 30° from a point on the ground. On retreating 100 meters, the measure of the angle of elevation is found to be of 15°. Find the height of the tower.

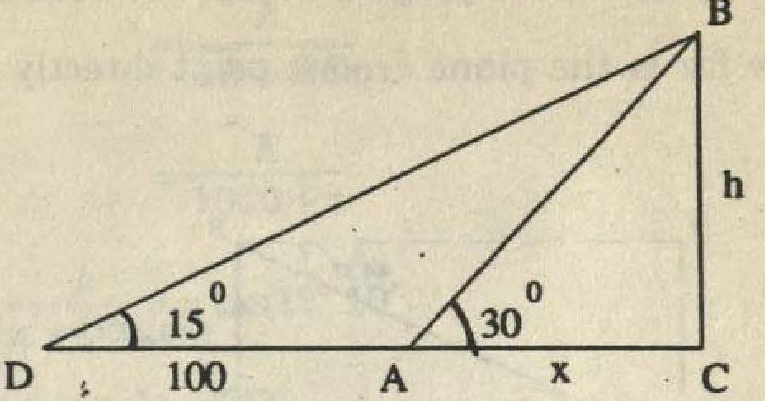


Fig. 11.8

Solution

Let the points *A* and *D* be the positions of the observer before and after the retreat. Height of the tower is denoted by '*h*'.

from the figure

$$\tan 30 = \frac{h}{x} \qquad \ldots \qquad (1)$$

$$\frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$\tan 15^o = \frac{h}{x+100} \qquad \ldots \qquad (2)$$

from (1) and (2) $\dfrac{\tan 30^o}{\tan 15^o} = 2.1547$

Hence $x = \dfrac{100}{1.1547} = 86.6m.$

Exercise 11.2

1. Measure of the angle of elevation of the top of a flagstaff observed from a point 200 meters from its

foot is 30°. Find its height.

2. Find the measure of the angle of elevation of the top of a tree 400 meters high, when observed from a point 250 meters away from the foot of the base.

3. How far is a man from the foot of a tower 150 meters high, if the measure of the angle of elevation of its top as obserrved by him is $40^o\ 30'$?

4. The measure of the angle of elevation of the top of a tower is 67°. What is the measure of the elevation of a point $\frac{2}{3}$ way up the tower from the same point where an observer stands?

5. A ladder 20 meters long reaches to a distance of 6 meters from the top of a pole. At the foot of the ladder the measure of the angle of elevation of the top of the pole is 80°. Find the height of the pole.

6. From a point on the ground the measure of angle of elevation of the top of a tower is 30°. One walking 100 meters towards the tower the measure of the angle is found to be of 45°. Find the height of the tower.

7. From the top of a hill 200 meters the measure of the angle of depression of the top and the bottom of a tree are $21^o\ 35'$ and $60^o\ 16'$ respectively. Find the height of the tree.

8. On walking 300 meters towards a tower in a horizontal line through its base, the measure of the angle of elevation of the top changes from 30° to 60°. Find the height of the tower.

9. Two men on the opposite sides of a tower observe that the measures of the angles of elevation of the tower as observed by them separately are 15° and 25° respectively. If the height of the tower is 150 meters, find the distance between the observers.

10. The shadow of a building is 220 meters long when the measure of the angle of elevation of the sun is 35°. Find the height of the building.

11. From a lookout tower 100 meters high, a man observes from a point 10 meters below the top of a tower that the measure of the angles of elevation and depression of the top and the bottom of a tree are $10^{\circ} 24'$ and $70^{\circ} 40'$ respectively. Base of the tree and that of the tower being at the same level. Find the height of the tree.

12. A pilot flying at a height of 5000 meters observer that the measure of the angle of depression of a tower is $20^{\circ} 24'$. How far is the plane from the tower?

13. The measure of the anglesof depression of the top of

a tower as observed from the top and the foot of a tree 100 meters high are 15° and 52° respectively. Find the height of the tower.

14. The measure of the angle of elevation of the top of a tower as observed from a point is 20° 24′. If the tower is 150 meters high, how far is the point from the foot of the tower?

15. Find the measure of the angle of elevation of the sun if a tower 300 meters high casts shadow 450 meter long.

16. From the top of a hill 2000 meters high the measure of the angle of depression of a point on the nearer shore of a river is 50° and the measure of angle of depression of a point directly across the river is 70°. Find the width of the river. How far is the river from the foot of the hill?

17. The measure of the angle of elevation of the top of a cliff is 25°. On walking 100 meters straight towards the cliff, the measure of the angle of elevation of the top is 48°. Find the height of the cliff.

18. The measure of the angle of elevation of a kite is 35°. The string of the kite is 340 meters long. If the sag in the string is 10 meters, find the height of the kite.

19. A ladder 20 meters reaches the distance of 20

meters, from the top of a building. At the foot of the ladder the measure of the angle of elevation of the top of the building is 60°. Find the height of the building.

20. From two points A and B 50 meters apart and in the line with a tree, the measures of the angles of elevation of the top of the tree are 30° and 40° respectively. Find the height of the tree.

11.3. The Law of Sines

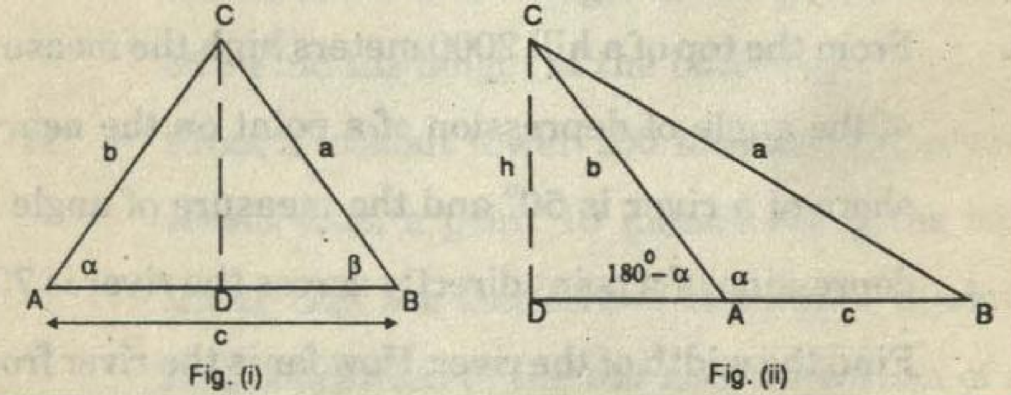


Fig. 11.9

Given the triangle ABC.

In Fig. (i) $\alpha < \frac{\pi}{2}$ and in fig (ii) $\alpha > \frac{\pi}{2}$.

Draw a perpendicular from point C to the opposite side AB such that it meets $\overline{AB}$ at D.

$m\,\overline{CD} = h$ in both cases.

In right triangle BDC fig. (i)

$$\sin \beta = \frac{h}{a}$$

so $h = a \sin\beta$ ... (1)

and in right triangle *ADC*

$$\sin\alpha = \frac{h}{b}$$

so $h = b\sin\alpha$ ... (2)

In fig. (ii), In right triangle BDC

$$\frac{h}{b} = \sin(\pi - \alpha) = \sin\alpha$$

or $h = b\sin\alpha$ (the same as (2))

from (1) and (2)

$$a\sin\beta = b\sin\alpha$$

$$\text{or } \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \quad \cdots \quad (3)$$

Similarly if we draw a perpendicular from point *A* to the side *BC* and proceeding as before we shall have

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \quad \cdots \quad (4)$$

from (3) and (4)

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

which is called the sine Law.

In otherwords, in any triangle the length of the sides are proportional to the sines of measures of the angles opposite to those sides.

Example 1

For a triangle ABC, given $a = 10$, $b = 15$, $\beta = 50^\circ$, find α.

Solution

Using the law of sines

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$$

$$\text{So } \sin\alpha=\frac{a}{b}\sin\beta$$

$$=\frac{10}{15}\sin 50^\circ$$

$$=.5107$$

$$\text{Hence } \alpha=30^\circ\ 42'\ 37''$$

Example 2

Given $\alpha = 40^\circ$, $a = 12$, $b = 7$, find β

Solution

Using the law

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}$$

$$\sin\beta=\frac{b}{a}\sin\alpha$$

$$=\frac{7}{12}\sin 40^\circ$$

$$=.3750$$

Therefore $\beta = 22^\circ\ 1'\ 18''$

Example 3

Given $\alpha = 60^\circ$, $\beta = 75^\circ$, $a = 16$, find c.

Solution

$$\alpha+\beta+\gamma=180^\circ$$

$\gamma = 45^\circ$

Using the law $\dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma}$

$$c = \frac{a}{\sin\alpha}\sin\gamma$$

$$= \frac{16}{\sin 60^\circ}\sin 45^\circ = 13.06$$

Exercise 11.3

In any triangle *ABC*

1. $b = 24, c = 16$, find the ratio of $\sin\beta$ to $\sin\gamma$
2. $a = 20, c = 32, \gamma = 70^\circ$, find α
3. $a = 3\ , b = 7, \beta = 85^\circ$, find $\sin\alpha$
4. $\alpha = 60^\circ, \beta = 45^\circ$, find the ratio of b to c
5. $a = 5, c = 6, \alpha = 45^\circ$, find $\sin\gamma$
6. $a = 20\sqrt{3}, \alpha = 75^\circ, \gamma = 60^\circ$, find c
7. $a = 211.3, \alpha = 48^\circ 16', \gamma = 71^\circ 38'$, find
8. $a = 18, \alpha = 47^\circ, \beta = 102^\circ$, find c
9. $a = 475, \alpha = 72^\circ 15', \gamma = 43^\circ 30'$, find b
10. $b = 82, \beta = 57^\circ, \gamma = 70^\circ$, find a
11. Two shore batteries at *A* and *B*, 840 meters apart are firing at a target *C*. The measure of angle *ABC* is 80° and the measure of angle *BAC* is 70°. Find the measures of the distances *AC* and *BC* to two decimal places.

12. **A town 'B' is 15 kilometer due north of a town 'A'. The road from A to B runs north 27° east to C, then north 34° west to B. Find the distance by road from town A to B.**

11.4 The Law Of Cosines

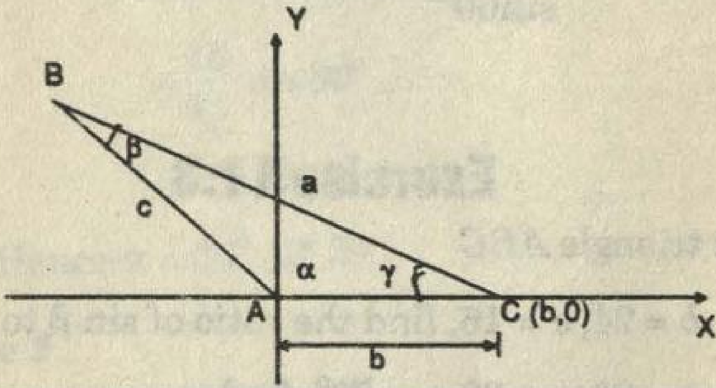


Fig. 11.10

By choosing a rectangular coordinate system so that the angle A of the triangle ABC is in the standard position with vertex Ç at $(b,0)$. The co-ordinates of B are $(c\cos\alpha, c\sin\alpha)$ and $c\,(b, 0)$

$$|\overline{BC}| = \sqrt{(c\cos\alpha - b)^2 + (c\sin\alpha - 0)^2}$$

$$= \sqrt{c^2\cos^2\alpha - 2b\,c\cos\alpha + b^2 + c^2\sin^2\alpha}$$

$$= \sqrt{c^2(\cos^2\alpha + \sin^2\alpha) + b^2 - 2bc\cos\alpha}$$

$$= \sqrt{b^2 + c^2 - 2bc\cos\alpha}$$

But $|\overline{BC}|$ is the length of the side $\overline{BC}$ of the triangle ABC s.t $|\overline{BC}| = a$

Therefore $a = \sqrt{b^2 + c^2 - 2bc\cos\alpha}$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos\alpha$$

Similarly we obtain

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

These formulae can also be expressed as

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc};$$

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ac};$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}.$$

These formulae are helpful, when all the measures of sides of a triangle are given. In this case we can find the measures of the angles of any triangle by using these formulae. We observe if any angle involved is a right angle, the law of cosines reduces to the Pythagorean Theorem.

Example 1

In triangle ABC

$b = 5, c = 8, \alpha = 60^\circ,$ find a

Solution

To find a, use law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$= 25 + 64 - 2(5)(8)\cdot\frac{1}{2}$$

$$= 49$$

$$a = 7$$

Example 2

for triangle *ABC*,

$$a = 9, b = 11, c = 14, find\alpha$$

Solution

To find α, use.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{121 + 196 - 81}{2.11.14}$$

$$= \frac{236}{308} = .7662$$

Therefore, $\alpha = 39^{\circ}\ 58'\ 59''$

Example 3

A body is acted upon by two forces 30 and 40 pounds respectively making an angle of measure $32^{\circ}\ 35'$ with each other. Find the magnitude of the resultant of the forces.

Solution

The forces are represented by the sides of parallelogram

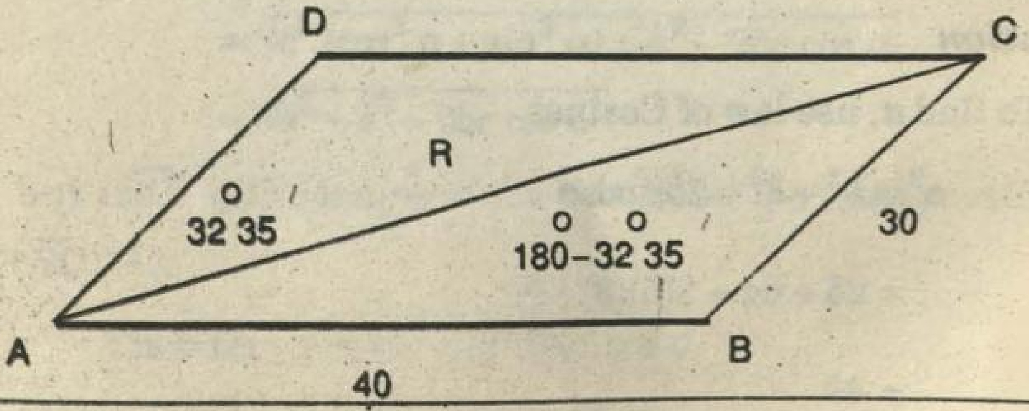


Fig. 11.11

ABCD. The resultant is represented by *R* and is given as

$$R^2 = (40)^2 + (30)^2 - 2(40)(30)\cos(180^o - 32^o\,35')$$

$$= 4522.26$$

Hence $R = 67.25$

Exercise 11.4.

In any triangle *ABC*

1. $a = 7, c = 9, \beta = 112^\circ$, find b
2. $a = 56, c = 30, \alpha = 35^\circ$, find b
3. $b = 25, c = 37, \alpha = 65^\circ$, find a
4. $a = 212, c = 135, \beta = 37^o\,15'$, find b
5. $a = 16, b = 17, \gamma = 25^\circ$, find c
6. $a = 44, b = 55, \gamma = 114^\circ$, find c
7. $a = 13, b = 10, c = 17$, find α and β
8. $a = 21, c = 34, \beta = 70^o\,50'$, find b
9. $a = 21.4, c = 34.1, \beta = 72^o\,34'$, find b
10. Two men start walking at the same time from a cross road, both walking at 4 km/hour. The roads make an angle of measure 80° with each other. How far apart they will be at the end of two hours?
11. Three villages *P*,*Q* and *R* are connected by straight roads. Measure $\overline{PQ}$ is 6 kilometer and the measure $\overline{QR}$ is 9 kilometers. The measure of the angle between *PQ* and *QR* is 120°. Find the distance be-

tween P and R.

12. Two points A and B are at distances 55 and 32 meters respectively from a point P. The measure of angle between AP and BP is 37°. Find the distance between B and A.

13. Find the cosine of the smallest measure of an angle of a triangle with 12, 13 and 14 meters as the measures of its sides.

14. Two ships at sea are approaching each other. A light house keeper observes that one ship is 2080 meters and the other is 2713 meters away from the light house. They attend on angle of measure $110^o 50'$ with vertex at the light house. How far apart are the two ships from each other?

11.5. The Law of Tangents

By the law of sines, in any triangle ABC

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = \text{D (say)}$$

Where a, b, c are the measures of the sides and α, β, γ are the measures of the angles of the triangle opposite to the sides a, b, c respectively.

$$a = \text{D}\sin\alpha \text{ and } b = \text{D}\sin\beta$$

$$a + b = \text{D}(\sin\alpha + \sin\beta) \quad \ldots \quad (1)$$

$$a - b = \text{D}(\sin\alpha - \sin\beta) \quad \ldots \quad (2)$$

from (1) and (2)

$$\frac{a+b}{a-b}=\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}$$

$$=\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$=\frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)} \quad \ldots \quad \text{(i)}$$

$$\text{Similarly } \frac{b+c}{b-c}=\frac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)} \quad \ldots \quad \text{(ii)}$$

$$\text{and } \frac{c+a}{c-a}=\frac{\tan\frac{1}{2}(\gamma+\alpha)}{\tan\frac{1}{2}(\gamma-\alpha)} \quad \ldots \quad \text{(iii)}$$

These three formulae are known as the Laws of Tangents. If in particular '*b*' is greater than *a* in (i), then the formula can be changed as

$$\frac{b+a}{b-a}=\frac{\tan\frac{1}{2}(\beta+\alpha)}{\tan\frac{1}{2}(\beta-\alpha)}.$$

It may be observed that the interchange of *a* and *b* also changes the positions of α and β. Similar positions hold for other formulae.

11.6 The Half Angle Formulae

We can deduce formulae by using the law of Cosines. These formulae are of great help to find the measures of all angles of a triangle when the measures of its sides are given.

Let a, b, c be the measures of the sides of a triangle. Let

$$s = \frac{1}{2}(a + b + c)$$

Using the law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos\alpha$$

Solving for cos α

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{But } \cos\alpha = 2\cos^2\frac{\alpha}{2} - 1$$

Therefore,

$$2\cos^2\frac{\alpha}{2} - 1 = \frac{b^2 + c^2 - a^2}{2bc}$$

$$2\cos^2\frac{\alpha}{2} = \frac{b^2 + c^2 - a^2}{2bc} - 1$$

$$= \frac{(b + c)^2 - a^2}{2bc}$$

$$= \frac{(b + c + a)(b + c - a)}{2bc}$$

$$b + c - a = 2s - 2a = 2(s - a)$$

so $$\cos^2\frac{\alpha}{2} = \frac{s(s - a)}{bc}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{s(s-a)}{bc}}$$

As $\frac{\alpha}{2}$ is the measure of an acute angle, the value of $\cos\frac{\alpha}{2}$ will be positive.

$$\text{Hence } \cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}} \quad \ldots \quad (1)$$

Similarly we can prove that

$$\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

$$\text{and} \quad \cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\text{Also} \quad \cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$$

$$\text{then} \quad 2\sin^2\frac{\alpha}{2} = \frac{(a+b-c)\,(a-b+c)}{2bc}$$

$$a + b - c = 2s - 2c$$

$$a - b + c = 2s - 2b$$

$$2\sin^2\frac{\alpha}{2} = \frac{(2s-2b)\,(2s-2c)}{2bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{(s-b)\,(s-c)}{bc}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{(s-b)\,(s-c)}{bc}}$$

As $\frac{\alpha}{2}$ is a measure of an acute angle, therefore $\sin\frac{\alpha}{2}$ is always positive.

Hence $\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ ... (2)

Similarly

$$\sin\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}} \text{ and } \sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

from (1) and (2)

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$= \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly $\tan\frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

and $\tan\frac{\gamma}{2} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}}$

We have $\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Multiply and divide the right hand member by $\sqrt{s-a}$ to get

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}$$

$$= \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

replacing $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ by r to get

$$\tan\frac{\alpha}{2} = \frac{r}{s-a}$$

The corresponding values for $\tan\frac{\beta}{2}$ and $\tan\frac{\gamma}{2}$ are

$$\tan\frac{\beta}{2} = \frac{r}{s-b} \text{ and } \tan\frac{\gamma}{2} = \frac{r}{s-c}$$

11.7 Solution Of Oblique Triangles

Triangles other than right triangles are called oblique triangles. The measures of the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of a triangle ABC shall be denoted by a, b and c respectively. The measures of the interior angles A, B and C of the triangle will be denoted by α, β and γ respectively. Measures of three sides and the three angles are called the parts of triangles. The problem of finding the solution of a triangle consists of finding the three of these parts when the other three parts are given. These given parts must include at least measure of one side.

There are following three cases in which three parts are given for solving a triangle:

(i) Measure of one side and the measures of two angles.

(ii) Measures of two sides and measure of the angle included by them.

(iii) Measures of the three sides.

(iv) Measures of two sides and measure of an angle opposite to one of them.

(This case is outside the scope of this book). We shall deal with the first three cases only for solving the triangles.

Type I: When measure of one side and measures of two angles are given

We shall use the law of sines to solve such triangles.

Example 1

Solve the triangle with given data

$$a = 850, \alpha = 65^\circ, \beta = 40^\circ$$

Solution

$$\gamma = 180^\circ - (\alpha + \beta)$$

$$= 180^\circ - 105^\circ$$

$$= 75^\circ$$

As $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

So $b = \dfrac{a}{\sin\alpha}.\sin\beta$

$$= \frac{850}{\sin 65^\circ}.\sin 40^\circ$$

$$= 602.85$$

To find c, we use

$$c = \frac{a}{\sin\alpha}\sin\gamma$$

$$= \frac{850}{\sin 65^\circ}.\sin 75^\circ$$

$$= 905.91$$

Type II: When measures of two sides and measure of the angle included by them are given

In this case, we shall use the law of cosines to find the measure of third side. When this is done, we can find the measure of one angle by using the law of sines. The third angle may be found by the formula

$$\alpha + \beta + \gamma = 180^\circ$$

We can also apply the law of tangents to solve the triangle when two sides and included angle are given.

The method shall be explained in the following examples:

Example 2

Solve the triangle with given data

$$a = 45, b = 34, \gamma = 52^\circ$$

Solution

To find c, using the law of cosines

$$\begin{aligned} c^2 &= a^2 + b^2 - 2\,ab \cos\gamma \\ &= (45)^2 + (34)^2 - 2\,(45)\,(34) \cos 52^\circ \\ &= 2025 + 1156 - (3060)\,(0.6157) \\ &= 2025 + 1156 - 1884 \\ &= 1297 \end{aligned}$$

So $c = 36.01$

To find α, use

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

$$\sin\alpha = \frac{a}{c}\sin\gamma$$

$$= \frac{45}{36.01}\sin 52^o$$

$$= .9847$$

$$\alpha = 79^o\ 59' = 80^o \text{ (app)}.$$

$$\alpha + \gamma = 52^o + 80^o = 132^o$$

$$\text{Hence } \beta = 180^o - 132^o = 48^o$$

Example 3

Solve the triangle *ABC* in which

$$a = 925,\ c = 432,\ \beta = 42^o\ 30'$$

Solution

Choosing the law of tangents which contains the given sides:

$$\frac{a-c}{a+c} = \frac{\tan\frac{1}{2}(\alpha-\gamma)}{\tan\frac{1}{2}(\alpha+\gamma)}$$

$$\alpha + \gamma = 180^o - \beta = 137^0\ 30'$$

$$\frac{1}{2}(\alpha+\gamma) = 68^0\ 45'$$

$$a - c = 493$$

$$a + c = 1360$$

$$\tan\frac{1}{2}(\alpha-\gamma) = \frac{493}{1360}\tan 68^o\ 45'$$

$$= .9322$$

Therefore $\frac{1}{2}(\alpha-\gamma)=42^{o}\,59'$

$\alpha-\gamma=85^{o}\,58'$

$\alpha+\gamma=137^{o}\,30'$

By addition $\alpha=111^{o}\,44'$

By substraction $\gamma=25^{o}\,46'$

To find b, we use the law of sines

$$\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

$$b=\frac{c}{\sin\gamma}\sin\beta$$

$$=\frac{432}{\sin 25^{o}\,46'}.\sin 42^{o}\,30'$$

$$=671.38$$

We have found all the missing parts for we had:

Given	found out
$a=925$	$\alpha=111^{o}\,44'$
$c=432$	$\gamma=25^{o}\,46'$
$\beta=42^{o}\,30'$	$b=671.38$

To find the results calculator has been used. The results are correct to two places of decimal.

Type III. When the measures of the sides of a triangle are given

In this case we can apply the law of cosines, but the computations became difficult when bigger numbers are involved. So we may apply the half angle formulae for solving triangles.

Example 4

Solve the triangle with the given data:

$a = 75, b = 55, c = 50.$

Solution

Given $a = 75, b = 55, c = 50$

Here $s = \frac{1}{2}(a + b + c)$

$= \frac{1}{2}(180) = 90$

So $s - a = 15$

$s - b = 35$

$s - c = 40$

Using half angle formula

$$\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{90(15)}{(55)(50)}}$$

$$= .700649$$

$$\frac{\alpha}{2} = 45^\circ\, 31'$$

$$\alpha = 91^\circ\, 2'$$

Also $\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}$

$= .9165$

$\frac{\beta}{2} = 23^\circ\ 35'$

Hence $\beta = 47^\circ\ 10'$

$\alpha + \beta = 138^\circ\ 12'$

$\gamma = 180^\circ - 138^\circ\ 12'$

$= 41^\circ\ 48'$

Exercise 11.5.

Solve the triangle *ABC* with given data:

1. $c = 4, \alpha = 70^\circ, \gamma = 42^\circ$
2. $a = 464, \beta = 102^\circ, \gamma = 23^\circ$
3. $b = 85, \beta = 57^\circ\ 15', \gamma = 78^\circ\ 18'$
4. $b = 56.8, \alpha = 79^\circ\ 31', \beta = 44^\circ\ 24'$
5. $b = 34.54, \alpha = 62^\circ\ 11', \beta = 63^\circ\ 22'$

Solve the triangle *ABC* with given data by using the 'Law of Tangents':

6. $a = 48, b = 32, \gamma = 57^\circ$
7. $b = 12.5, c = 23, \alpha = 38^\circ\ 20'$
8. $b = 35, c = 37, \alpha = 23^\circ\ 25'$
9. $a = 88, b = 48, \gamma = 75^\circ\ 51'$
10. $a = 168, c = 319, \beta = 110^\circ\ 22'$

Find the largest angle of the triangle *ABC* where:

11. $a = 224, b = 380, c = 340$

12. $a = 374, b = 514, c = 425$

Solve the triangle ABC where:

13. $a = 74, b = 52, c = 47$

14. $a = 7, b = 9, c = 7$

15. $a = 2.3, b = 1.5, c = 2.7$

16. $a = 49, b = 35, c = 44$

17. $a = 57.8, b = 34.4, c = 72.2$

Solve the triangle ABC by using half angle formulas for cosines and check the result by applying the half angle formulas for sines.

18. $a = 413, b = 214, c = 375$

19. $a = 217, b = 315, c = 412$

20. $a = 4.76, b = 3.24, c = 6.00$

21. $a = 2.43, b = 4.13, c = 5.12$

22. $a = 73.3, b = 63.5, c = 84.5$

23. $a = 271.6, b = 325.8, c = 385.4$

24. $a = 7.241, b = 4.837, c = 6.495$

11.8. Area of a triangle

(a) From the elementary geometry we know that the area of a triangle is equal to one half the product of measure of the base and measure of the altitude.

Let the given triangle be ABC in which measure $\overline{AB} = c$. Let measure of the altitude be h.

If Δ denotes the one of the triangle ABC, then using the

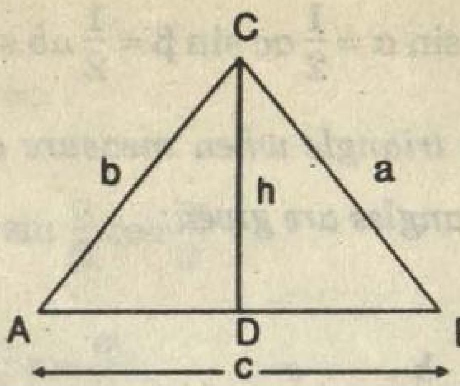


Fig. 11.12

above figure

$$\Delta = \frac{1}{2}\text{(measure of the base)} \cdot \text{(measure of the altitude)}$$

$$= \frac{1}{2}c \cdot h$$

But $\sin\alpha = \frac{h}{b}$ and $\sin\beta = \frac{h}{a}$

Therefore $h = b\sin\alpha$ and $h = a\sin\beta$.

Hence $\Delta = \frac{1}{2}bc\sin\alpha$.

Also $\Delta = \frac{1}{2}ac\sin\beta$

$$= \frac{1}{2}ab\sin\gamma.$$

This shows that the area of a triangle is one half the product of the measures of two sides and the sine of the measure of the angle included between these two sides.

This area of a triangle with given measures of two sides and measure of included angle is given by the formula.

$$\Delta = \frac{1}{2}bc \sin \alpha = \frac{1}{2} ac \sin \beta = \frac{1}{2} ab \sin \gamma \qquad (1)$$

(b) *Area of a triangle when measure of one side and the measures of two angles are given:*

We have

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$\text{So } c = \frac{b}{\sin \beta} \sin \gamma$$

In formulae (1) replace the values of the second letter in terms of the first from the law of sines.

put $c = \frac{b}{\sin \beta} \sin\gamma$ in the formula

$$\Delta = \frac{1}{2} bc \sin \alpha$$

$$\text{Thus } \Delta = \frac{1}{2} b \frac{b}{\sin \beta} \sin \gamma \sin \alpha$$

$$= \frac{1}{2} b^2 \frac{\sin \gamma \sin\alpha}{\sin \beta}$$

$$\text{Similarly } \Delta = \frac{1}{2} a^2 \frac{\sin \beta \sin \gamma}{\sin \alpha}$$

$$= \frac{1}{2} c^2 \frac{\sin\alpha \sin\beta}{\sin\gamma}$$

These formulae give the area of a triangle when measure of one side and the measures of two angles are given.

(c) *Area of a triangle when measures of all the sides of a triangle are given:*

We know that

$$\Delta = \frac{1}{2} bc \sin\alpha$$

$$= \frac{1}{2} bc \, .2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2}$$

$$= bc \sin\frac{\alpha}{2} \cos\frac{\alpha}{2}$$

Using half angle formulae

$$\Delta = bc \sqrt{\frac{(s-b)(s-c)}{bc}} . \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

Thus the area of a triangle with given measures of its sides is

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

This formula is known as Hero's formula for finding the area of a triangle with given measures of its sides.

We shall illustrate the methods for finding the areas of triangles by using the above mentioned formulae.

Example 1

Find the area of a triangle in which

$$a = 756.52, c = 543.35, \beta = 47^\circ \, 17'$$

to two places of decimal.

Solution

$$\Delta = \frac{1}{2} ac \sin\beta$$

$$=\frac{1}{2}(756.52)(543.35)\sin 47^o 17'$$

$$= 151004.66 \text{ sq. units.}$$

Example 2

Find the area of the triangle in which

$$a = 22.24, \beta = 32^o 15', \gamma = 65^o 37'.$$

Solution

$$\Delta = \frac{1}{2}a^2\frac{\sin\beta\sin\gamma}{\sin\alpha}.$$

$$\alpha + \beta + \gamma = 180^o$$

$$\alpha = 180^o - (\beta + \gamma)$$

$$= 180^0 - (32^o 15' + 65^o 37')$$

$$= 82^o 8'$$

$$\text{Therefore } \Delta = \frac{1}{2}(22.24)^2\ \frac{(\sin 32^o 15')(sin\ 65^o 37')}{\sin 82^o 8'}$$

$$= 121.34 \text{ sq. units.}$$

Example 3

Find the area of a triangle with given data

$$a = 52.345, b = 29.572, c = 43.751$$

Solution

$$\text{Here } s = \frac{1}{2}(a + b + c)$$

$$= \frac{1}{2}[52.345 + 29.572 + 43.751]$$

$$= 62.8325$$

$s - a = 10.4905$

$s - b = 33.2595$

$s - c = 19.0815$

$\Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$

$= \sqrt{(62.8325)\,(10.4905)\,(33.2595)\,(19.0815)}$

$= 646.77672 = 646.78$ sq. units.

Exercise 11.6

Find the area of the triangle ABC with the given data:

1. $a = 37, b = 45, \gamma = 30^o\ 50'$
2. $a = 315, c = 425, \beta = 70^o\ 45'$
3. $b = 414, c = 485, \alpha = 49^o\ 47'$
4. $a = 276, b = 315, c = 524$
5. $a = 212, b = 415, c = 321$
6. $a = 2.34, b = 3.15, c = 4.25$
7. $a = 32, \beta = 47^o\ 24', \gamma = 70^o\ 16'$
8. $b = 47, \alpha = 60^0\ 25', \gamma = 41^o\ 35'$
9. $c = 57, \alpha = 23^o\ 24', \beta = 71^o\ 36'$
10. $a = 35, \beta = 60^o, \gamma = 45^o$
11. $a = 45, \mathrm{b} = 37, \gamma = 75^o$
12. $a = 24, \alpha = 37^o, \gamma = 63^o$
13. $a = 925, \mathrm{c} = 433, \beta = 42^o\ 10'$
14. $a = 92, b = 71, \alpha = 56^o\ 44'$

15. $b = 4.8, \beta = 83^\circ 42', \gamma = 37^\circ 12'$

11.9. Circles connected with Triangles

(a) Circumcircle

A circle passing through the vertices of any triangle is called the circumcircle associated with that triangle. The radius of this circle is called its circumradius. The centre of this circle is called its circumcentre. Measure of the circumradius is denoted by 'R'.

(b) In-circle

A circle drawn inside a triangle and touching its sides is called the incircle associated with the triangle. Its radius is called the inradius and its centre is called the incentre of the triangle. Measure of the inradius of a triangle is usually denoted by 'r'.

(c) Escribed circles

A circle which touches one side of a triangle externally and the other two sides internally when produced, is called escribed circle of the triangle associated with that side to which it touches externally. There are three such circles corresponding to three sides.

The measure of the radius of the circle opposite to the vertex A of the triangle ABC is denoted by r_1 and measures of the radii of the circles opposite to the vertices B and C are denoted by r_1 and r_2 respectively. The centres of these circles opposite to the vertices A, B and C are denoted by I_1, I_2 and I_3 respectively. The centres are also called excentres of the triangles.

The reader knows that:

(i) The circumcentre of a triangle is the point at which the right bisectars of its sides meet each other.

(ii) The incentre of a triangle is the point at which the internal bisectars of a triangle meet each other.

(iii) The ex-centre with respect to the vertex A is the point of intersection of the external bisectars of angles B and C and internal bisectar of angle A.

11.10 (a) To find R, the circumradius of a triangle ABC in terms of measures of a side and its opposite angle

We shall prove that

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}.$$

Let 'O' be the circumcentre of the triangle ABC. Join B and O and produce it to meet the circle D. Join C and D.

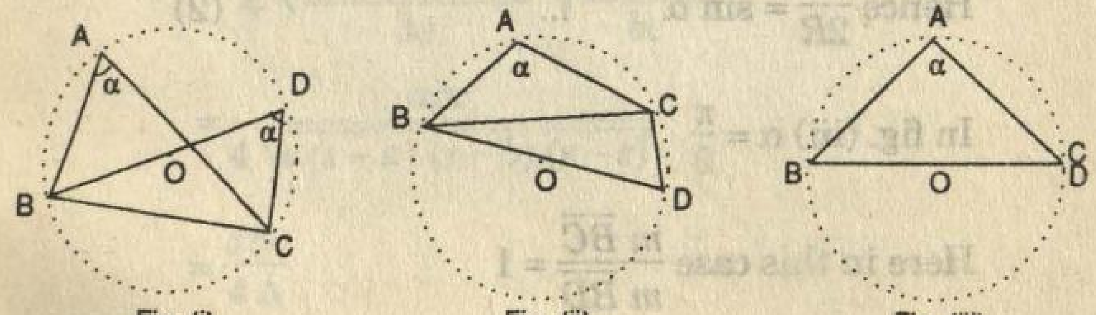


Fig. (i) Fig. (ii) Fig. (iii)

Fig. 11.13

Measure $\overline{BD}$ is a diameter of the circle

So $\overline{BD} = 2R$ where

R is the circumradius of the circle.

$m\,\overline{BC} = a$

In fig. (i) $m < BDC = \alpha < \frac{\pi}{2}$

In fig. (ii) $m < BDC = \pi - \alpha$

In fig. (iii) $m < BAC = \alpha = \frac{\pi}{2}$

In fig. (i)

$$\frac{m\,\overline{BC}}{m\,\overline{BD}} = \sin m < BDC$$

So $\frac{a}{2R} = \sin\alpha \quad \ldots \quad (1)$

In fig. (ii)

$$\frac{m\,\overline{BC}}{m\,\overline{BD}} = \sin m < BDC$$

$$= \sin(\pi - \alpha)$$

$$= \sin\alpha$$

Hence $\frac{a}{2R} = \sin\alpha \quad \ldots \quad (2)$

In fig. (iii) $\alpha = \frac{\pi}{2}$

Here in this case $\frac{m\,\overline{BC}}{m\,\overline{BD}} = 1$

$$\frac{m\,\overline{BC}}{m\,\overline{BD}} = 1 = \sin\frac{\pi}{2} = \sin\alpha$$

Hence $\frac{a}{2R} = \sin\alpha$ ... (3)

In all the cases $R = \frac{a}{2\sin\alpha}$

Similarly we can prove that

$$R = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

Thus in all the cases

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

(b) Circumradius in terms of the measurements of sides of a triangle ABC

Here $R = \frac{a}{2\sin\alpha}$

$$= \frac{a}{2.2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

Using half angle formulae for $\sin\frac{\alpha}{2}$ and $\cos\frac{\alpha}{2}$

$$R = \frac{a}{4\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}}$$

$$= \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{abc}{4\Delta}$$

Where $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

Example

Find R for the triangle measures of whose sides are 11, 12 and 13 meters.

Solution

Let $a = 11, b = 12$, and $c = 13$

$$R = \frac{abc}{4\,\Delta}$$

$$\Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$= \sqrt{18.7.6.5}$$

$$R = \frac{11.12.13}{4\,\sqrt{18.7.6.5}} = 6.98\,meters.$$

11.11. To find inradius 'r' for any triangle ABC

We shall prove

$$r = \frac{\Delta}{s}$$

Where $s = \frac{a+b+c}{2}$ and Δ is the area of the triangle.

Let the internal besectors of the angles of a triangle ABC meet at the point I, the incentre.

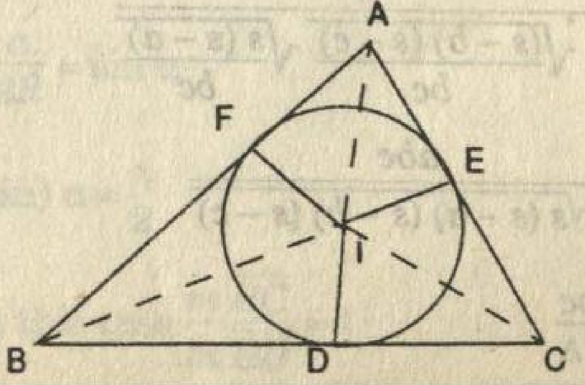


Fig. 11.14

Draw $\overline{ID}, \overline{IE}$ and $\overline{IF}$ perpendicular on the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$ respectively.

We have

$$m\,\overline{ID} = m\,\overline{IE} = m\,\overline{IF} = r$$

Joint the points I and A, I and B and I and C.

From the figure

$$\text{Area } \Delta ABC = \text{Area } \Delta BIC + \text{Area } \Delta CIA + \text{Area } \Delta BIA$$

$$= \frac{1}{2} ar + \frac{1}{2} br + \frac{1}{2} cr$$

$$= \frac{1}{2} r\,(a + b + c)$$

$$a + b + c = 2\,s$$

So $\text{Area } \Delta ABC = \frac{1}{2} r.\, 2s = rs$

Therefore $r = \dfrac{\text{Area } \Delta ABC}{s}$

If we write Δ for area of a triangle then

$$r = \frac{\Delta}{s}$$

Example

Prove that in any triangle ABC

$$r = 4\,R \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}$$

Solution

$$\text{R.H.S.} = 4\,R \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2}$$

$$= 4 \cdot \frac{abc}{4\,\Delta} \sqrt{\frac{(s-b)\,(s-c)}{bc}} \sqrt{\frac{(s-a)\,(s-c)}{ac}} \sqrt{\frac{(s-a)\,(s-b)}{ab}}$$

$$= \frac{1}{\Delta} (s-a)\,(s-b)\,(s-c)$$

Multiplying and divide right member by s then

$$\text{R.H.S.} = \frac{1}{\Delta} \frac{s\,(s-a)\,(s-b)\,(s-c)}{s}$$

$$= \frac{1}{s\,\Delta} \Delta^2 = \frac{\Delta}{s} = r = L.H.S.$$

$$\text{As } \Delta = \sqrt{s(s-a)\,(s-b)\,(s-c)}$$

11.12. To Find the Radii of e-circles of a triangle

We shall prove that

$$r_1 = \frac{\Delta}{s-a}$$

Let I_1, be the e-centre opposite to the vertex.

Let L, M, N be the points at which the e-circle touches the side $\overline{BC}$, externally and touches the sides $\overline{AB}$ and $\overline{AC}$ when produced respectively.

I_1L, I_1M and I_1N are perpendiculars to $\overline{BC}$, $\overline{AB}$ and $\overline{AC}$

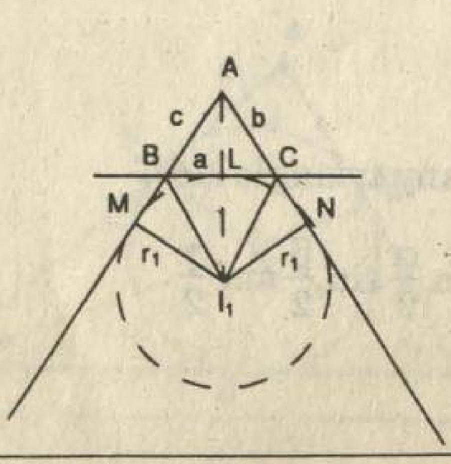


Fig. 11.15

(produced) respectively.

Join I_1, and A, I_1, and B, I_1, and C.

Then

$$m\,\overline{I_1L} = m\,\overline{I_2M} = m\,\overline{I_3N} = r_1$$

Clearly

$$\begin{aligned}\Delta ABC &= \Delta AI_1B + \Delta AI_1C - \Delta BI_1C \\ &= \frac{1}{2}cr_1 + \frac{1}{2}br_1 - \frac{1}{2}ar_1 \\ &= \frac{1}{2}r_1(c+b-a) \\ &= \frac{1}{2}r_1(2s-2a) \\ &= r_1(s-a)\end{aligned}$$

Therefore $r_1 = \dfrac{\Delta}{s-a}$

Where Δ stands for the area of triangle ABC.

Similarly $r_2 = \dfrac{\Delta}{s-b}$

$$r_3 = \frac{\Delta}{s-c}$$

Example I

Find R, r, r_1, r_2 and r_3 for the triangle with measure of the sides

$a = 18$, $b = 24$, $c = 30$ centimeters.

Solution

$$s = \frac{a+b+c}{2}$$

$$= 36$$

$$s-a = 18, s-b = 12, s-c = 6$$

$$\Delta = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$$

$$= \sqrt{36.18.12.6}$$

$$= 216$$

$$R = \frac{abc}{4\Delta} = \frac{18.24.30}{4.216}$$

$$= 15 \text{ centimeter.}$$

$$r = \frac{\Delta}{s} = \frac{216}{36} = 6 \text{ centimeter}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{216}{18} = 12 \text{ centimeter}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{216}{12} = 18 \text{ centimeter}$$

$$r_3 = \frac{\Delta}{s-c} = \frac{216}{6} = 36 \text{ centimeter}$$

Example 2

Prove that for any equilateral triangle

$$r : R : r_1 = 1 : 2 : 3.$$

Solution

Let the measure of a side of an equilateral triangle be a.

$$s = \frac{a+a+a}{2} = \frac{3a}{2}$$

As for an equilateral triangle measures of the sides are the same.

Area of the triangle is given by

$$\Delta = \sqrt{s\,(s-a)^3} = \sqrt{\frac{3a}{2} \cdot \frac{a^3}{8}}$$

$$= \frac{\sqrt{3}}{4} a^2$$

$$R = \frac{a \cdot b \cdot c}{4\Delta} = \frac{a^3}{4\,\frac{\sqrt{3}}{4} a^2},$$

$$= \frac{a}{\sqrt{3}}$$

$$r = \frac{\Delta}{s} = \frac{\sqrt{3}}{4} a^2 \cdot \frac{2}{a} = \frac{a}{2\sqrt{3}}.$$

$$r_1 = \frac{\Delta}{s-a} = \frac{\sqrt{3}}{4} a^2 \cdot \frac{2}{a} = \frac{\sqrt{3}\,a}{2}$$

$$\text{Now } r : R : r_1 = \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}\,a}{2}$$

$$= 1 : 2 : 3$$

Example 3

Find the area of the inscribed circle of the triangle measures of whose sides are 7, 8 and 9.

Solution

$$s = \frac{7+8+9}{2} = 12$$

Area of the triangle $= \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$

$$= 26.83$$

$$r = \frac{\Delta}{s} = \frac{26.83}{12} = 2.36$$

Area of the inscribed circle

$$= \pi r^2 = \frac{22}{7}(2.36)^2 = 17.5 \text{ Sq. units.}$$

Exercise 11.7

1. If the measures of the sides of a triangle are 53, 65 and 60 meters, find the radii of its escribed circles.
2. If the measures of the sides of a triangle are 25, 55 and 70 meters, find R and r for the triangle.
3. Show that $\sqrt{r\, r_1\, r_2\, r_3} = \Delta = \frac{1}{2} bc \sin\alpha$
4. If e-radii of any triangle ABC are in $A.P$, then show that $(a-b)(s-c) = (b-c)(s-a)$

for any triangle ABC, prove that

5. $R\, r(\sin\alpha + \sin\beta + \sin\gamma) = \Delta$
6. $(s-a)(s-b)\cot\frac{\gamma}{2} = \Delta$
7. $\frac{abc}{4s}(\sin\alpha + \sin\beta + \sin\gamma) = \Delta$
8. $\sin\alpha = \frac{2\Delta}{bc}$
9. $\Delta = s^2 \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
10. $\Delta = \frac{abc}{s}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

11. $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\frac{s}{r}$

12. $r_1=s\tan\frac{\alpha}{2}$

13. $r_1+r_2+r_3-r=4R$

14. $r_1=4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

15. $r_2=4R\sin\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}$

16. $r_3=4R\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}$

17. $r_1r_2+r_2r_3+r_3r_1=s^2$

18. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$

19. $\Delta=r^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

20. $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$

21. Find the area of the inscribed circle of the triangle with measures of the sides 34, 35, 36 decimeters.

22. The measures of the sides of a triangle are 20, 25 and 30 decimeters. Find the radius of the escribed circle opposite to the largest side.

23. Find the area of the circumcircle of the triangle measures of whose sides are 50, 60 and 65 meters.

24. Prove that for any triangle ABC

$$r = (s - a)\tan\frac{\alpha}{2} = (s - b)\tan\frac{\beta}{2} = (s - c)\tan\frac{\gamma}{2}$$

25. For any triangle ABC

$$(r_1 + r_{2)}\tan\frac{\gamma}{2} = c$$

Chapter 12

Inverse Trigonometric Functions and Trigonometric Equations

12.1. Inverse Trigonometric Functions

Let R denote the set of all real numbers, then the sine function is defined as

$$\{ (x, y) \mid y = \sin x, x \in R, -1 \leq y \leq 1 \}$$

(i) for each element in its domain, there is a unique element in its range,

(ii) this function is periodic i.e. for each element in the range, there are infinite elements in its domain.

The difficulty in defining its inverse arises due to its periodicity. Any how we shall overcome this difficulty by restricting its domain.

The inverse of the sine function is defined as

$$\{ (x, y) \mid x = \sin y, -1 \leq x \leq 1, y \in R \}$$

The inverse of the sine function is usually denoted by

$y = \sin^{-1} x$

or $y = \text{arc} \sin x$

Using this notation, we state the above definition

as $\quad y = \sin^{-1}x$ if and only if $x = \sin y$

for $\quad -1 \le x \le 1$ and $y \in R$

One should not confuse $\sin^{-1}x$ with $(\sin x)^{-1}$ which means $\dfrac{1}{\sin x}$

We draw the graph of $y = \sin^{-1}x$ to show that it is not a function.

Consider the graph of $y = \sin^{-1}x$ (or $x = \sin y$) shown in the figure 12.1 and also note the solution of the equation

x	y	x	y
0	$-\pi$	0	0
$-\frac{1}{2}$	$-\frac{5\pi}{6}$	$\frac{1}{2}$	$\frac{\pi}{6}$
$-\frac{1}{\sqrt{2}}$	$-\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$
$-\frac{\sqrt{3}}{2}$	$-\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
-1	$-\frac{\pi}{2}$	1	$\frac{\pi}{2}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{2\pi}{3}$
$-\frac{1}{\sqrt{2}}$	$-\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{3\pi}{4}$
$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{5\pi}{6}$
		0	π

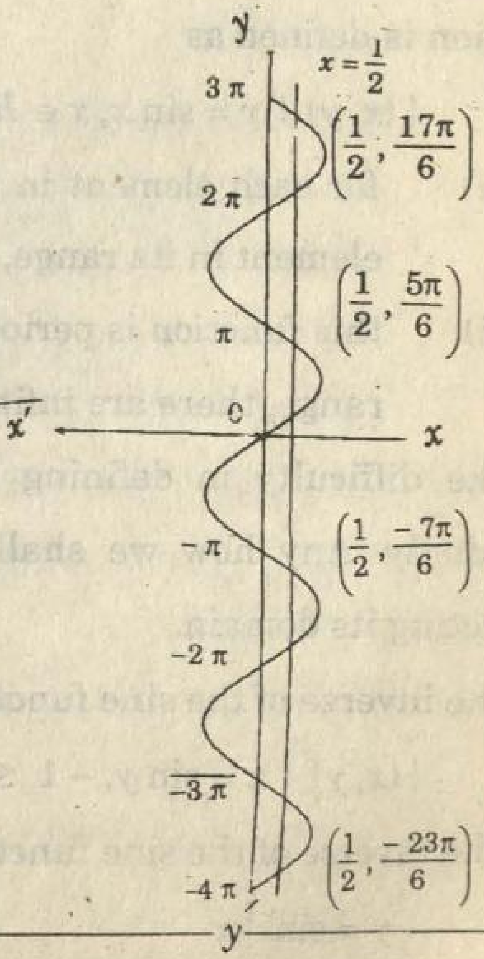


Fig. 12.1

$$y = \sin^{-1}\frac{1}{2} \quad \left(\text{or } \frac{1}{2} = \sin y\right)$$

Each value of y such that

$$y = (2\pi)\,n + \frac{\pi}{6}$$

$$\text{or } y = (2\pi)\,n + \frac{5\pi}{6}$$

(where $n \in Z$, the set of integers) satisfies the equation

$$y = \sin^{-1}\frac{1}{2}$$

The line $x = \frac{1}{2}$ is also drawn in the figure 12.1 to visualize the solution.

It should be noted that the solution set consists of the second components of the points of inter section of the line $x = \frac{1}{2}$ and the graph of $y = \sin^{-1}x$.

Some points of intersection are shown in the figure, the v-coordinates of these points are

$$-\frac{23\pi}{6}, -\frac{19\pi}{6}, -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

So we conclude that

(i) the inverse of the sine function maps each $x \in [-1, 1]$ to infinite set of numbers $y \in R$,

(ii) It is a relation but not a function.

The bold part of the graph of $y = \sin x$

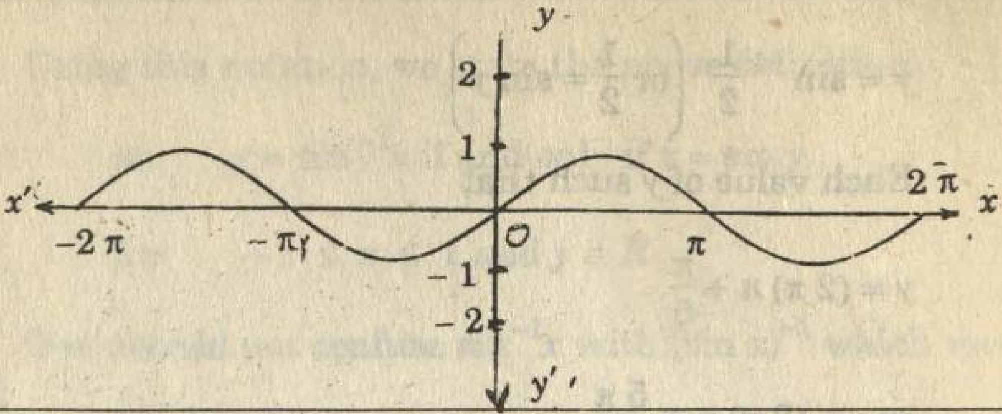


Fig. 12.2

for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

has the property that each value of y corresponds to a unique value of x. This is so because Sin x is increasing throughout the interval $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$ and hence it is a one – one function. The restriction of the sine function to the domain $\left\{x \mid -\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$ is denoted by the symbol Sin and is called the principal sine function. The value of Sin at x is written as Sin x.

The principal sine function is defined by $y = \text{Sin } x$, where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $y \in [-1, 1]$.

Definition

The inverse sine function, denoted by Sin^{-1} or arc Sin, is the inverse of the Principal sine function and is defined by

$$y = \text{Sin}^{-1}x \text{ if and only if } x = \text{Sin } y$$

for $-1 \le x \le 1$ and $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$

The graph of $y = \text{Sin}^{-1}x$ verifies the fact that it is a function. (see the figure 12.3).

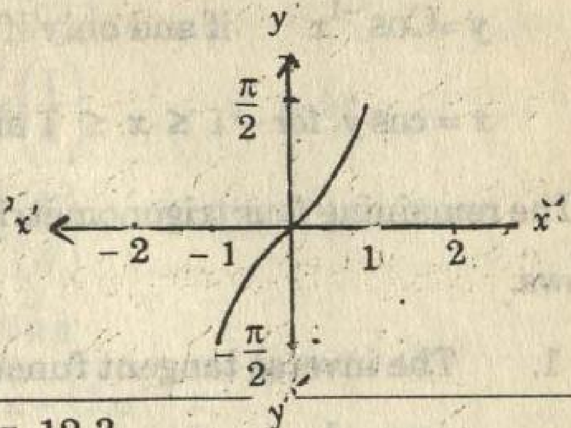


Fig. 12.3

The situation is similar in the case of other trigonometric functions can similarly be treated. If the domain of cosine function is restricted to the interval $[0, \pi]$, we obtain a restricted cosine function whose inverse function exists and is denoted by arc cosine or Cos^{-1}. The value of Cos^{-1} (or arc Cosine) at x is denoted by $\text{Cos}^{-1}x$ (or arc Cos x).

The graph of $y = \text{Cos}^{-1}x$ can be obtained from the graph of $y = \text{Cos}\, x$ by the reflection about the line $y = x$. (see the figure 12.4)

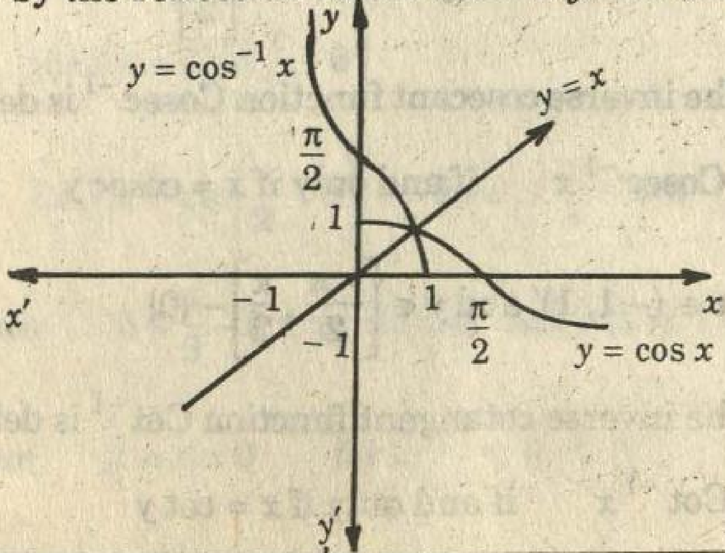


Fig. 12.4

Definition

The inverse cosine function denoted by Cos^{-1} or arc Cos is the inverse of the Principal cosine function and is defined by

$$y = \text{Cos}^{-1}x \qquad \text{if and only if}$$

$$x = \cos y \ \text{ for } -1 \le x \le 1 \text{ and } 0 \le y \le \pi.$$

The remaining four trigonometric functions are defined as follows:

1. The inverse tangent function Tan^{-1} is defined by

$$y = \text{Tan}^{-1}x \qquad \text{if and only if } x = \tan y$$

$$\text{for } x \in (-\infty, \infty) \text{ and } y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

2. The inverse secant function Sec^{-1} is defined by

$$y = \text{Sec}^{-1}x \qquad \text{if and only if } x = \sec y$$

$$\text{for } x \in (-1, 1)' \text{ and } y \in [0, \pi] - \left[\frac{\pi}{2}\right].$$

3. The inverse cosecant function Cosec^{-1} is defined by

$$y = \text{Cosec}^{-1}x \qquad \text{if and only if } x = \text{cosec}\, y$$

$$\text{for } x \in (-1, 1)' \text{ and } y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

4. The inverse cotangent function Cot^{-1} is defined by

$$y = \text{Cot}^{-1}x \qquad \text{if and only if } x = \cot y$$

$$\text{for } x \in (-\infty, \infty) \text{ and } y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

12.2 Now we write some relations involving inverse functions.

(i) $\text{Cosec}^{-1} x = \overset{-1}{Sin}\left(\dfrac{1}{x}\right) \qquad (x \neq 0)$

(ii) $\text{Sec}^{-1} x = \overset{-1}{\text{Cos}}\left(\dfrac{1}{x}\right) \qquad (x \neq 0)$

(iii) $\text{Cot}^{-1} x = \overset{-1}{\text{Tan}}\left(\dfrac{1}{x}\right) \qquad (x \neq 0)$

(iv) $\overset{-1}{\text{Sin}}(-x) = -\overset{-1}{\text{Sin}}\, x$

(v) $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1} x.$

(vi) $\text{Tan}^{-1}(-x) = -\overset{-1}{\text{Tan}}\, x$

(vii) $\text{Sin}^{-1} x = \dfrac{\pi}{2} - \text{Cos}^{-1} x$

To prove (vii), let $\theta = \dfrac{\pi}{2} - Cos^{-1} x$

i.e., $\text{Cos}^{-1} x = \dfrac{\pi}{2} - \theta$

then $x = Cos\left(\dfrac{\pi}{2} - \theta\right)$

i.e. $x = \cos\left(\dfrac{\pi}{2} - \theta\right) \quad$ for $0 \le \dfrac{\pi}{2} - \theta \le \pi$

Since $\quad 0 \le \dfrac{\pi}{2} - \theta \le \pi$, we can verify that $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

Thus $\quad x = \sin\theta \quad$ for $\dfrac{-\pi}{2} \le \theta \le \dfrac{\pi}{2}$

i.e. $x = \text{Sin}\,\theta$

Hence by definition $\theta = \text{Sin}^{-1} x$

$$\text{Hence } \operatorname{Sin}^{-1}x=\frac{\pi}{2}-\operatorname{Cos}^{-1}x$$

For convenience, the domains and ranges of the principal trigonometric functions and their inverses are shown in the following table:

Function defined by	***Domian***	***Range***
$y=\operatorname{Sin}x$	$\frac{-\pi}{2}\le x\le\frac{\pi}{2}$	$-1\le y\le 1$
$y=\operatorname{Sin}^{-1}x$	$-1\le x\le 1$	$\frac{-\pi}{2}\le y\le\frac{\pi}{2}$
$y=\operatorname{Cos}x$	$0\le x\le\pi$	$-1\le y\le 1$
$y=\operatorname{Cos}^{-1}x$	$-1\le x\le 1$	$0\le y\le\pi$
$y=\operatorname{Tan}x$	$\frac{-\pi}{2}< x<\frac{\pi}{2}$	$y\in R$ (the set of real numbers)
$y=\operatorname{Tan}^{-1}x$	$x\in R$	$\frac{-\pi}{2}< y<\frac{\pi}{2}$
$y=\operatorname{Cosec}x$	$-\frac{\pi}{2}\le x<0, 0<x\le\frac{\pi}{2}$ or $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$	$y\le-1, y\ge 1$ or $y\in R-(-1, 1)=(-1, 1)'$
$y=\operatorname{Cosec}^{-1}x$	$x\le-1, x\ge 1$ or $x\in(-1, 1)'$	$-\frac{\pi}{2}\le y<0, 0<y\le\frac{\pi}{2}$ or $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$
$y=\operatorname{Sec}x$	$0\le x<\frac{\pi}{2}, \frac{\pi}{2}<x\le\pi$ or $x\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$	$y\le-1, y\ge 1$ or $y\in(-1, 1)'$
$y=\operatorname{Sec}^{-1}x$	$x\le-1, x\ge 1$ or $x\in(-1, 1)'$	$y\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$
$y=\operatorname{Cot}x$	$x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$	$y\in R$

Function defined by	Domian	Range
$y = \mathrm{Cot}^{-1} x$	$x \in \mathbb{R}$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Example 1

Evaluate;

(i) $\mathrm{Cos}^{-1}\left(-\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution

(i) Let $y = \mathrm{Cos}^{-1}\left(-\frac{1}{2}\right)$

then $\mathrm{Cos}\, y = -\frac{1}{2}$

i.e. $\cos y = -\frac{1}{2}$ for $y \in [0, \pi]$

$\therefore\ \mathrm{Cos}^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

(ii) Let $y = \cos^{-1}\left(-\frac{1}{2}\right)$

Then $\cos y = -\frac{1}{2}$ for $y \in \mathbb{R}$

But $y = (2\pi)\, n + \frac{2\pi}{3} \left(\cos\left(2\pi n + \frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}\right)$

or $y = (2\pi)\, n + \frac{4\pi}{3} \left(\cos\left(2\pi n + \frac{4\pi}{3}\right) = \cos\frac{4\pi}{3} = -\frac{1}{2}\right)$

Where $n \in \mathbb{Z}$ (the set of integers)

$\therefore$ the solution set of the equation;

$\cos y = -\frac{1}{2}$ is $\left\{2\pi n + \frac{2\pi}{3}\right\} \cup \left\{2\pi n + \frac{4\pi}{3}\right\}$

Where $n \in Z$

Thus $\cos^{-1}\left(-\frac{1}{2}\right) = \left\{2\pi n + \frac{2\pi}{3}\right\} \cup \left\{2\pi n + \frac{4\pi}{3}\right\}$ Where $n \in Z$

Example 2

Evaluate

(i) $\mathrm{Tan}^{-1}(-\sqrt{3})$ (ii) $\tan^{-1}(-\sqrt{3})$.

Solution

(i) Let $y = \mathrm{Tan}^{-1}(-\sqrt{3})$

Then $\mathrm{Tan}\, y = -\sqrt{3}$

i.e $\tan y = -\sqrt{3}$ for $\frac{-\pi}{2} < y < \frac{\pi}{2}$

But $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$ for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \mathrm{Tan}^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

(ii) Let $\tan^{-1}(-\sqrt{3}) = y$

Then $\tan y = -\sqrt{3}$ for $y \in R - R_1$

Where $R_1 = \left\{t \mid t = k\pi + \frac{\pi}{2}\right\}, k \in Z$

i.e $\tan y = -\sqrt{3}$ for y is a real number such that $y \neq k\pi + \frac{\pi}{2}$

But $y = n_1\pi + \left(\frac{2\pi}{3}\right)$ $\left[\tan\left(n_1\pi + \frac{2\pi}{3}\right) = \tan\frac{2\pi}{3} = -\sqrt{3}\right]$

or $y = n_2\pi + \left(-\frac{\pi}{3}\right)$ $\left[\tan\left(n_2\pi + \left(-\frac{\pi}{3}\right)\right) = \tan\left(\frac{-\pi}{3}\right) = -\sqrt{3}\right]$

where $n_1, n_2 \in Z$

$\therefore$ The solution set of the equation $\tan y = -\sqrt{3}$ is

$$\left\{n_1 \pi + \frac{2\pi}{3}\right\} \cup \left\{n_2 \pi + (-\frac{\pi}{3})\right\} \text{ where } n_1, n_2 \in Z$$

$$\text{Thus } \tan^{-1}(-\sqrt{3}) = \left\{n_1 \pi + \frac{2\pi}{3}\right\} \cup \left\{n_2 \pi + (-\frac{\pi}{3})\right\}$$

$$= \left\{n\ \ \pi + \frac{2\pi}{3}\right\}$$

Example 3

Evaluate;

(i) $\tan\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ (ii) $\tan\left(\text{Cos}^{-1}\left(-\frac{1}{2}\right)\right)$

(iii) $\text{Tan}\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ (iv) $\text{Tan}\left(\text{Cos}^{-1}\left(-\frac{1}{2}\right)\right)$

Solution

(i) $\cos^{-1}\left(-\frac{1}{2}\right) = \left\{2\pi n + \frac{2\pi}{3}\right\} \cup \left\{2\pi n + \frac{4\pi}{3}\right\}$

(See example I (ii))

$$\text{But } \tan\left(2\pi n + \frac{2\pi}{3}\right) = \tan\frac{2\pi}{3} = -\sqrt{3} \text{ and}$$

$$\tan\left(2\pi n + \frac{4\pi}{3}\right) = \tan\frac{4\pi}{3} = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\therefore \tan\left(\cos^{-1}\left(-\frac{1}{2}\right)\right) = \{-\sqrt{3}, \sqrt{3}\}.$$

(ii) $\text{Cos}^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ (See example I (i))

But $\tan\frac{2\pi}{3} = -\sqrt{3}$

$$\therefore \tan\left(\mathrm{Cos}^{-1}\left(-\frac{1}{2}\right)\right) = -\sqrt{3}$$

(iii) $\mathrm{Tan}\left(2\pi n + \frac{2\pi}{3}\right)$ and $\mathrm{Tan}\left(2\pi n + \frac{4\pi}{3}\right)$ are not defined even if $n = 0$ because $\frac{2\pi}{3}, \frac{4\pi}{3} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \mathrm{Tan}\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ is not defined.

(iv) $\mathrm{Tan}\left(\mathrm{Cos}^{-1}\left(-\frac{1}{2}\right)\right) = \mathrm{Tan}\left(\frac{2\pi}{3}\right)$

But $\mathrm{Tan}\left(\frac{2\pi}{3}\right)$ is not defined as $\frac{2\pi}{3} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \mathrm{Tan}\left(\mathrm{Cos}^{-1}\left(-\frac{1}{2}\right)\right)$ is not defined.

Example 4

Evaluate;

(i) $\mathrm{arc\,Sin}\left(\sin\frac{12\pi}{5}\right)$ (ii) $\mathrm{arc\,sin}\left(\sin\frac{12\pi}{5}\right)$

(iii) $\mathrm{arc\,Cos}\left(\cos\frac{29\pi}{7}\right)$ (iv) $\mathrm{arc\,cos}\left(\cos\frac{29\pi}{7}\right)$

Solution

(i) As $\frac{12\pi}{5} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \mathrm{arc\,Sin}\left(\sin\frac{12\pi}{5}\right) \neq \frac{12\pi}{5}$$

But $\sin\left(\frac{12\pi}{5}\right) = \sin\left(2\pi + \frac{2\pi}{5}\right) = \sin\frac{2\pi}{5}$

$$\text{As } \frac{2\pi}{5} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{Thus arc Sin}\left(\sin\frac{12\pi}{5}\right) = \text{arc Sin}\left(\sin\frac{2\pi}{5}\right) = \frac{2\pi}{5}$$

(ii) $$\text{arc sin}\left(\sin\frac{12\pi}{5}\right) = \frac{12\pi}{5}$$

(iii) $$\text{arc Cos}\left(\cos\frac{29\pi}{7}\right) \neq \frac{29\pi}{7} \qquad \left(\because \frac{29\pi}{7} \notin [0, \pi]\right)$$

$$\text{But }\cos\left(\frac{29\pi}{7}\right) = \cos\left(4\pi + \frac{\pi}{7}\right) = \cos\frac{\pi}{7}$$

$$\therefore\ \text{arc Cos}\left(\cos\frac{29\pi}{7}\right) = \text{arc Cos}\left(\cos\frac{\pi}{7}\right) = \frac{\pi}{7}$$

(iv) $$\text{arc cos}\left(\cos\frac{29\pi}{7}\right) = \frac{29\pi}{7}.$$

Example 5

Evaluate $\tan\left(\text{arc Sin}\,\frac{4}{5}\right)$

Solution

Let $\text{arc Sin}\,\frac{4}{5} = y$

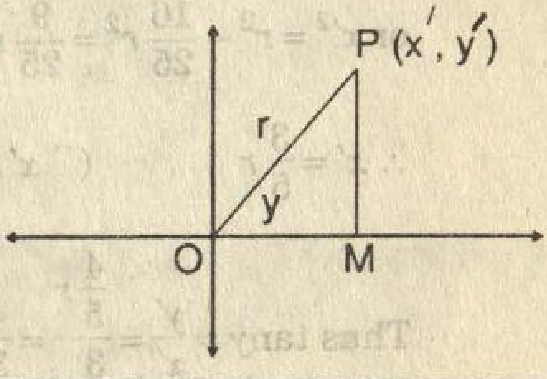


Fig. 12.5

Then $\text{Sin}\, y = \frac{4}{5}$

i.e $\sin y = \frac{4}{5}$ for $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

We want to find tan y where $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \leq \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. As sin y is positive, it means that y lies in the first quadrant i.e. $y \in \left[0, \frac{\pi}{2}\right]$ and tan y is positive.

From the right angled triangle OMP, We have

$$x'^2 + y'^2 = r^2 \quad \ldots \qquad \text{(i)}$$

$$\text{But} \quad \frac{y'}{r} = \frac{4}{5} \qquad \left(\because \sin y = \frac{4}{5}\right)$$

$$\therefore \quad y' = \frac{4}{5} r$$

Putting $y' = \frac{4}{5} r$ in (i), we obtain

$$x'^2 + \left(\frac{4}{5} r\right)^2 = r^2$$

$$\text{or } x'^2 = r^2 - \frac{16}{25} r^2 = \frac{9}{25} r^2$$

$$\therefore x' = \frac{3}{5} r \qquad (\because x' \text{ is positive})$$

$$\text{Thus } \tan y = \frac{y'}{x'} = \frac{\frac{4}{5} r}{\frac{3}{5} r} = \frac{4}{3}$$

$$\text{Hence } \tan\left(\text{arc Sin} \frac{4}{5}\right) = \frac{4}{3}$$

Example 6

Express tan (arc Sin x) as an algebraic function of x if $-1 < x < 1$.

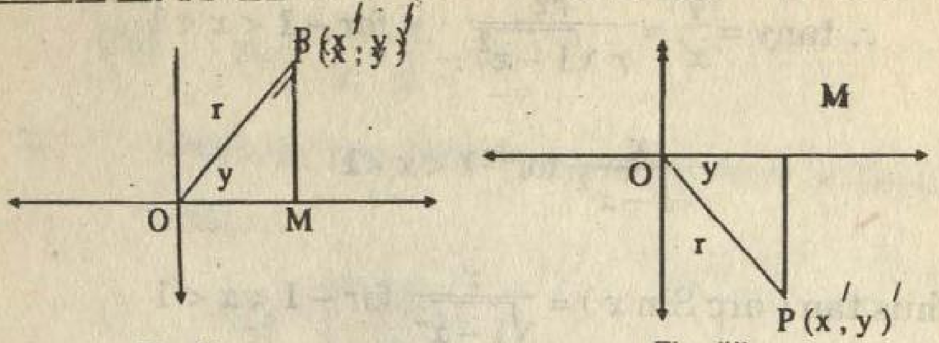


Fig. (i) Fig. (ii)

Fig. 12.6

Solution

Let arc Sin $x = y$

Then $\quad \text{Sin}\, y = x$

i.e. $\sin y = x$ for $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $x \in [-1, 1]$.

We want to find $\tan y$ in terms of x where $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

(i) If x is positive, then $y \in \left(0, \frac{\pi}{2}\right)$

(ii) If x is negative, then $y \in \left(\frac{-\pi}{2}, 0\right)$

The triangles in Fig (i) and Fig (ii) are shown for both the cases. From either of the triangles, we have

$$x'^2 + y'^2 = r^2$$

$$x'^2 + (rx)^2 = r^2 \qquad \left(\because \ \sin y = \frac{y'}{r} = x \ \text{ i.e } y' = rx\right)$$

$$x'^2 = r^2 - r^2x^2 = r^2(1 - x^2)$$

$$x' = r\sqrt{1 - x^2} \qquad (\because \ x' \text{ is positive in both cases})$$

$$\therefore \tan y = \frac{y'}{x'} = \frac{rx}{r\sqrt{1-x^2}} \qquad \text{for } -1 < x < 1$$

$$= \frac{x}{\sqrt{1-x^2}} \text{ for } -1 < x < 1$$

Thus $\tan(\text{arc Sin } x) = \dfrac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$

Note: $\dfrac{x}{\sqrt{1-x^2}}$ is positive when x is positive and is negative when x is negative.

Example 7

Express $\cos(\sin^{-1}x)$ as an algebraic function of x if $-1 \le x \le 1$.

Solution

Let $\sin^{-1} x = y$

Then $\sin y = x$ for $y \in R$

But $\cos y = \pm\sqrt{1-\sin^2 y}$

$$= \pm\sqrt{1-x^2} \quad (\because \sqrt{1-x^2} \text{ is real if } -1 \le x \le 1)$$

$\therefore \cos(\sin^{-1} x) = \pm\sqrt{1-x^2}$ for $-1 \le x \le 1$

Exercise 12.1

1. Find x if

(i) $\text{Sin}^{-1}\dfrac{1}{2} = \dfrac{\pi}{2} - x$

(ii) $\text{Cos}^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{2} - \text{Sin}^{-1} x$

(iii) $\sin^{-1}\frac{1}{2}=\frac{\pi}{2}-x$

2. Evaluate the following expressions without using tables.

(i) $\text{Sin}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(ii) $\text{Cos}^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(iii) $\text{Cosec}^{-1}(-\sqrt{2})$

(iv) $\text{Cosec}^{-1}\left(-\frac{2}{\sqrt{2}}\right)$

(v) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

3. Evaluate

(i) $\text{Sin}\left(\frac{\pi}{2}-\text{Cos}^{-1}\frac{4}{5}\right)$

(ii) $\sin\left(\text{arc Cos}\frac{4}{5}+\pi\right)$

(iii) $\cos\left(\frac{\pi}{2}-\sin^{-1}\frac{5}{13}\right)$

(iv) $\cos(\pi-\text{Sin}^{-1}x)$

4. Find the relation between α and β if

$$\text{Sin}^{-1}\alpha=\frac{\pi}{2}-\text{Cos}^{-1}\beta$$

5. Show that $\cos(\text{Sin}^{-1}x-\text{Sin}^{-1}y)=\sqrt{(1-x^2)(1-y^2)}+xy$

6. Show that $\cos(2\,\text{Sin}^{-1}x)=1-2x^2$

7. Show that

(i) $\tan(\pi + \mathrm{Tan}^{-1}x) = x$

(ii) $\tan\left(\pi + \mathrm{Cot}^{-1}x\right) = \frac{1}{x}$

8. Compute the following expressions

(i) $\sin\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)$

(ii) $\sin\left(\mathrm{Tan}^{-1}\frac{1}{\sqrt{3}}\right)$

(iii) $\sin\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

(iv) $\cos^{-1}\left(\tan\left(-\frac{5\pi}{4}\right)\right)$

(v) $\mathrm{Tan}^{-1}\left(\tan\frac{3\pi}{4}\right)$

(vi) $\mathrm{Sin}^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right)$

(vii) $\mathrm{Cos}^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$

(viii) $\tan\left(\mathrm{Cos}^{-1}\left(-\frac{4}{5}\right)\right)$

(ix) $\mathrm{Tan}\left(\mathrm{Cos}^{-1}\left(-\frac{4}{5}\right)\right)$

(x) $\mathrm{Sin}\left(\mathrm{Cot}^{-1}\frac{3}{4}\right)$

(xi) $\cos\left(\mathrm{Sec}^{-1}\left(\frac{-3}{12}\right)\right)$

(xii) $\sin\left(\mathrm{Sin}^{-1}\frac{12}{13} + \mathrm{Sin}^{-1}\frac{4}{5}\right)$

(xiii) $\tan\left(\mathrm{Tan}^{-1}\frac{1}{2}+\mathrm{Tan}^{-1}\left(-\frac{2}{3}\right)\right)$

(xiv) $\cos\left(\mathrm{Tan}^{-1}\frac{3}{4}-\mathrm{Tan}^{-1}\left(-\frac{1}{7}\right)\right)$

9. Verify that

(i) $2\,\mathrm{Tan}^{-1}\frac{1}{2}+\mathrm{Tan}^{-1}\left(-\frac{1}{7}\right)=\frac{\pi}{4}$

(ii) $2\,\mathrm{Tan}^{-1}\frac{1}{3}+\mathrm{Tan}^{-1}\frac{1}{7}=\frac{\pi}{4}$

(iii) $\mathrm{Sin}^{-1}\frac{77}{85}-\mathrm{Sin}^{-1}\frac{3}{5}=\mathrm{Cos}^{-1}\frac{15}{17}$

(iv) $\mathrm{Tan}^{-1}\frac{5}{6}+\mathrm{Tan}^{-1}\frac{1}{11}=\frac{\pi}{4}$

10. Show that

(i) $\mathrm{Tan}^{-1}x-\mathrm{Tan}^{-1}y=\mathrm{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$

(ii) $\mathrm{Tan}^{-1}x+\mathrm{Tan}^{-1}y=\mathrm{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

(iii) $\mathrm{Sin}^{-1}x+\mathrm{Sin}^{-1}y=\mathrm{Sin}^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

(iv) $\mathrm{Sin}^{-1}x-\mathrm{Sin}^{-1}y=\mathrm{Sin}^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$

(v) $\mathrm{Cos}^{-1}x+\mathrm{Cos}^{-1}y=\mathrm{Cos}^{-1}\left(xy-\sqrt{(1-x^2)(1-y^2)}\right)$

(vi) $\mathrm{Cos}^{-1}x-\mathrm{Cos}^{-1}y=\mathrm{Cos}^{-1}\left(xy+\sqrt{(1-x^2)(1-y^2)}\right)$

(vii) $\mathrm{Tan}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\mathrm{Cos}^{-1}x$ where $0\le x\le 1$

(viii) $\mathrm{Cos}^{-1}x=\mathrm{Sin}^{-1}\sqrt{1-x^2}$ if $0\le x\le 1$

(ix) $\mathrm{Tan}^{-1}\frac{1+x}{1-x}-\mathrm{Tan}^{-1}x=\frac{\pi}{4}$, $|x|<1$

11. Express $\frac{\pi}{4} - \text{Tan}^{-1}\frac{1}{11}$ as a single inverse Tangent.

12. The angles α, β, θ and lengths a, h, x are shown in the diagram. Show that

$$\theta = \text{Tan}^{-1}\left(\frac{hx}{x^2 + a^2 + ah}\right)$$

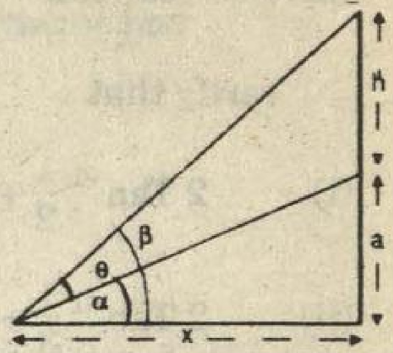


Fig. 12.7

$$\theta = \text{Sin}^{-1}\left(\frac{hx}{\left(\sqrt{(a+h)^2 + x^2}\right)\left(\sqrt{a^2 + x^2}\right)}\right)$$

12.3. Trigonometric Equations

In this section we shall use radian measure for the variables in the trigonometric expressions. Our aim will be to find all those values of the variable which satisfy the trigonometric equations. We shall first determine those values of the variable which belong to $[0, 2\pi]$ or $\left[\frac{-\pi}{2}, \frac{3\pi}{2}\right]$ and then write the general solution. The method of solving Trigonometric equations will be illustrated with the help of the following solved examples.

Example **1**

Solve the equation $2\sin x - \sqrt{3} = 0$

Solution

$$2\sin x - \sqrt{3} = 0$$

$$\text{or } \sin x = \frac{\sqrt{3}}{2}$$

The values of $x \in [0, 2\pi]$ which satisfy the above equation are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

$\therefore$ all values of x of the form $2\pi n + \frac{\pi}{3}$ or $2\pi n + 2\frac{\pi}{3}$, where $n \in Z$ satisfy the equation $\sin x = \frac{\sqrt{3}}{2}$

Thus the general solution set of the given equation is

$$\left\{2\pi n + \frac{\pi}{3}\right\} \cup \left\{2\pi n + \frac{2\pi}{3}\right\} \qquad \text{where } n \in Z$$

Example 2

Solve the equation $2 \cos 2x = \sqrt{3}$

Solution

$$2 \cos 2x = \sqrt{3}$$

$$\text{or } \cos 2x = \frac{\sqrt{3}}{2} \quad \dots \qquad (A)$$

Let $2x = \theta$, then the equation (A) becomes

$$\cos\theta = \frac{\sqrt{3}}{2} \quad \dots \qquad (B)$$

The values of $\theta \in [0, 2\pi]$ which satisfy the equation (B) are $\frac{\pi}{6}$ and $\frac{11\pi}{6}$

$$\text{Therefore } \theta = 2\pi n + \frac{\pi}{6}$$

$$\text{or } \theta = 2\pi n + \frac{11\pi}{6} \quad \text{where } n \in Z$$

satisfy the equation (B).

Now substituting $2x$ for θ, we have

$$2x = 2\pi n + \frac{\pi}{6} \text{ i.e. } x = \pi n + \frac{\pi}{12}$$

$$\text{or } 2x = 2\pi n + \frac{11\pi}{6} \text{ i.e. } x = \pi n + \frac{11\pi}{12}$$

Hence the general solution of the equation $2\cos 2x = \sqrt{3}$ is

$$\left\{\pi n + \frac{\pi}{12}\right\} \cup \left\{\pi n + \frac{11\pi}{12}\right\} \qquad \text{where } n \in Z$$

Example 3

Solve $\quad 2\sin x\cos x - \sin x = 0$

Solution

$$2\sin x\cos x - \sin x = 0 \quad \ldots \quad \text{(i)}$$

$$\text{or } \sin x\,(2\cos x - 1) = 0$$

Equating each factor to zero, we have

$$\sin x = 0 \quad \ldots \quad \text{(ii)}$$

$$\text{or } \cos x = \frac{1}{2} \quad \ldots \quad \text{(iii)}$$

The equation $\sin x = 0$ is satisfied by $0, \pi$ belonging to $[0, 2\pi]$.

$\therefore$ the solution set of (ii) is $\{2\pi n\} \cup \{2\pi n + \pi\}$ where $n \in Z$.

The values of $x \in [0, 2\pi]$ satisfying the equation $\cos x = \frac{1}{2}$

are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

$\therefore$ the solution set of (iii) is

$$\left\{2\pi n + \frac{\pi}{3}\right\} \cup \left\{2\pi n + \frac{5\pi}{3}\right\} \quad \text{Where } n \in Z$$

Thus the general solution of the equation (i) is

$$\{2\pi n\} \cup \{2\pi n + \pi\} \cup \left\{2\pi n + \frac{\pi}{3}\right\} \cup \left\{2n\pi + \frac{5\pi}{3}\right\}$$

Where $n \in Z$

Example 4

Solve $2\sin^2 x + 3\sin x - 2 = 0$

Solution

$$2\sin^2 x + 3\sin x - 2 = 0$$

The above equation is quadratic in $\sin x$,

$$\text{Also } \sin x = \frac{-3 \pm \sqrt{9+16}}{4}$$

$$= \frac{1}{2}, -2$$

Since $-1 \le \sin x \le 1$ i.e., $\sin x \in [-1, 1]$, it follows that

$\sin x = \frac{1}{2}$ has a solution but $\sin x = -2$ has no solution as

$-2 \notin [-1, 1]$.

The equation $\sin x = \frac{1}{2}$ is satisfied by $x = \frac{\pi}{6}, \frac{5\pi}{6} \in [0, 2\pi$

$\therefore$ its solution set is $\left\{2\pi n + \frac{\pi}{6}\right\} \cup \left\{2\pi n + 5\frac{\pi}{6}\right\}$ Where $n \in Z$

Thus the solution set of the given equation is

$$\left\{2\pi n + \frac{\pi}{6}\right\} \cup \left\{2\pi n + 5\frac{\pi}{6}\right\} \text{ Where } n \in Z$$

Example 5

Solve $2 \sin^2 2x - 3 \cos 2x - 3 = 0$

Solution

$$2 \sin^2 2x - 3 \cos 2x - 3 = 0 \quad \dots \quad \text{(i)}$$

The equation (i) can be written as

or $2 \cos^2 2x + 3 \cos 2x + 1 = 0.$

Using quadratic formula, we get

$$\cos 2x = \frac{-3 \pm \sqrt{9-8}}{4}$$

$$= -\frac{1}{2}, -1$$

The value of $2x \in [0, 2\pi]$ which satisfies the equation

$\cos 2x = -1$ is π

Therefore the solution of $\cos 2x = -1$ is

$$2x = 2\pi n + \pi$$

i.e. $x = \pi n + \frac{\pi}{2}$ where $n \in Z$

The values of $2x \in [0, 2\pi]$ which satisfy the equation

$\cos 2x = -\frac{1}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Therefore all values of $2x$ which satisfy the equation

$\cos 2x = -\frac{1}{2}$ are

$$2x = 2\pi n + \frac{2\pi}{3} \text{ i.e. } x = \pi n + \frac{\pi}{3}$$

$$\text{or } 2x = 2\pi n + \frac{4\pi}{3} \text{ i.e. } x = \pi n + \frac{2\pi}{3} \qquad \text{Where } n \in Z$$

Thus the general solution of the equation (i) is

$$\left\{n\pi + \frac{\pi}{2}\right\} \cup \left\{n\pi + \frac{\pi}{3}\right\} \cup \left\{n\pi + \frac{2\pi}{3}\right\} \qquad \text{where } n \in Z$$

Example 6

Solve the equation $4 \sin^2 x \cos x - \sin 2x = 0$

Solution

$$4 \sin^2 x \cos x - \sin 2x = 0 \quad \dots \quad \text{(i)}$$

The equation (i) can be written as

$(2 \sin x \cos x = \sin 2x)$

or $\sin 2x\,(2 \sin x - 1) = 0.$

$$\Rightarrow \sin 2x = 0 \quad \dots \quad \text{(ii)}$$

$$2 \sin x - 1 = 0 \text{ i.e. } \sin x = \frac{1}{2} \quad \dots \quad \text{(iii)}$$

The equation $\sin 2x = 0$ is satisfied by 0 and π belonging to $[0, 2\pi]$.

Therefore all values of $2x$ are of the form $2\pi n$ or $2\pi n + \pi$ satisfying the equation $\sin 2x = 0$.

So we have, $2x = 2\pi n \qquad$ i.e. $x = \pi n$

or $2x = 2\pi n + \pi \qquad$ i.e. $x = \pi n + \frac{\pi}{2}$ Where $n \in Z$

The solution set of (ii) is $\{\pi n\} \cup \left\{\pi n + \frac{\pi}{2}\right\}$. Where $n \in Z$

The values of x belonging to $[0, 2\pi]$ and satisfying $\sin x = \frac{1}{2}$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Therefore all values of x satisfying $\sin x = \frac{1}{2}$ are of the form $2\pi n + \frac{\pi}{6}$ or $2\pi n + \frac{5\pi}{6}$

So the solution set of (iii) is $\left\{2\pi n + \frac{\pi}{6}\right\} \cup \left\{2\pi n + \frac{5\pi}{6}\right\}$.

Where $n \in Z$

Hence the general solution set of (i) is

$$\{n\pi\} \cup \left\{n\pi + \frac{\pi}{2}\right\} \cup \left\{2\pi n + \frac{\pi}{6}\right\} \cup \left\{2\pi n + \frac{5\pi}{6}\right\}, n \in Z$$

Example 7

Solve $\sin 3\theta - \sin\theta = 0$.

Solution

We know that $\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

$\therefore\ \sin 3\theta - \sin\theta = 2\cos 2\theta \sin\theta$

The given equation is equivalent to

$2\cos 2\theta \sin\theta = 0$.

i.e. $\cos 2\theta = 0$

or $\sin\theta = 0$

All values of 2θ satisfying the equation $\cos 2\theta = 0$ are of the form $2\pi n + \frac{\pi}{2}$ or $2\pi n + \frac{3\pi}{2}$

$$\therefore\ \theta = \pi n + \frac{\pi}{4}$$

$$\text{or } \theta = \pi n + \frac{3\pi}{4}$$

So the solution set of $\cos 2\theta = 0$ is

$$\left\{\pi n + \frac{\pi}{4}\right\} \cup \left\{\pi n + \frac{3\pi}{4}\right\}.$$

The solution set of $\sin\theta = 0$ is $\{n\pi\}$ (see example 6)

Thus the genersl solution set of the given equation is

$$\{n\pi\} \cup \left\{\pi n + \frac{\pi}{4}\right\} \cup \left\{\pi n + \frac{3\pi}{4}\right\} \qquad \text{where } n \in Z$$

Alternative method

We know that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore$ the equation $\sin 3\theta - \sin\theta = 0$ can be written as

$$3\sin\theta - 4\sin^3\theta - \sin\theta = 0$$

$$\text{or } 2\sin\theta - 4\sin^3\theta = 0$$

which is equivalent to $\sin\theta\,(1 - 2\sin^2\theta) = 0$

Thus we have

$$\sin\theta = 0 \quad \ldots \qquad \text{(i)}$$

$$\text{or } 1 - 2\sin^2\theta = 0$$

which gives $\sin\theta = \pm\dfrac{1}{\sqrt{2}}$

The solution of $\sin\theta = 0$ is $\theta = n\pi$ where $n \in Z$

The solution of $\sin\theta = \dfrac{1}{\sqrt{2}}$ is $\theta = 2m\pi + \dfrac{\pi}{4}$

$$\text{or } \theta = 2m\pi + \frac{3\pi}{4}, \quad m \in Z$$

The solution of $\sin\theta = -\frac{1}{\sqrt{2}}$ is $\theta = 2m\pi + \frac{5\pi}{4}$

$$\text{or } \theta = 2m\pi + \frac{7\pi}{4}, m \in Z$$

But the values of $\theta = 2m\pi + \frac{\pi}{4}$ or $(2m+1)\pi + \frac{\pi}{4}$ can be written as $\theta = n\pi + \frac{\pi}{4}$ $(m, n \in Z)$

and the values of $\theta = 2m\pi + \frac{3\pi}{4}$ or $(2m+1)\pi + \frac{3\pi}{4}$ can be expressed as $\theta = n\pi + \frac{3\pi}{4}$, $(m, n \in Z)$

Therefore the solution set of $\sin 3\theta - \sin\theta = 0$ is

$$\{n\pi\} \cup \left\{n\pi + \frac{\pi}{4}\right\} \cup \left\{n\pi + \frac{3\pi}{4}\right\} \quad \text{where } n \in Z$$

Example 8

Solve the equation

$$\sqrt{3}\tan x - \sec x - 1 = 0$$

Solution

The given equation is

$$\sqrt{3}\tan x - \sec x - 1 = 0 \quad \ldots \quad (A)$$

which is equivalent to

$$\sqrt{3}\tan x = \sec x + 1 \quad \ldots \quad (B)$$

Squaring both sides of (B), we obtain

$$3\tan^2 x = \sec^2 x + 1 + 2\sec x$$

$$3(\sec^2 x - 1) = \sec^2 x + 1 + 2\sec x$$

On simplifying, we get

$$\sec^2 x - \sec x - 2 = 0$$

which is quadratic in sec x,

$$\therefore \sec x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$$= 2, -1$$

$$\text{i.e. } \cos x = \frac{1}{2} \text{ or } \cos x = -1$$

The values of x of the form $2\pi n + \frac{\pi}{3}$ or $2\pi n + \frac{5\pi}{3}$ where $n \in Z$ satisfy $\cos x = \frac{1}{2}$ and the equation $\cos x = -1$ is satisfied by the values of x of the form $2\pi n + \pi$ where $n \in Z$

But the equation (A) is not satisfied by the values of

$$x = 2\pi n + \frac{5\pi}{3} \qquad (n \in Z)$$

Therefore the solution set of (A) is

$$\{2n\pi + \pi\} \cup \left\{2\pi n + \frac{\pi}{3}\right\}, \qquad \text{where } n \in Z$$

Example 9

Solve the equations

(i) $4\text{Cos}^2 2x = 3$

(ii) $\text{Tan}^2 x + (\sqrt{3} - 1)\text{Tan}x - \sqrt{3} = 0$

Solution

(i) $4\text{Cos}^2 2x = 3$

which is equivalent to $\text{Cos}^2 2x = \frac{3}{4}$

or $\text{Cos } 2x = \pm \frac{\sqrt{3}}{2}$

Thus we have

$$\text{Cos } 2x = \frac{\sqrt{3}}{2} \qquad \text{(i)}$$

$$\text{or Cos } 2x = -\frac{\sqrt{3}}{2} \quad \ldots \qquad \text{(ii)}$$

We want to find the values of $2x$ from the interval $[0, \pi]$ satisfying the equations (i) and (ii)

So we have $2x = \frac{\pi}{6}$ or $2x = \frac{5\pi}{6}$

i.e. $x = \frac{\pi}{12}$ or $x = \frac{5\pi}{12}$

Therefore the solution set of the given equation is

$$\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}.$$

(ii) $\text{Tan}^2 x + (\sqrt{3} - 1)\,\text{Tan} x - \sqrt{3} = 0$ (A)

Using the quadratic formula, we get

$$\text{Tan } x = \frac{-(\sqrt{3} - 1) \pm (\sqrt{3} + 1)}{2}$$

$$= 1, -\sqrt{3}$$

The equation Tan $x = 1$ is satisfied by $x = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $x = -\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies the equation Tan $x = -\sqrt{3}$

Therefore the solution set of (A) is $\left\{-\frac{\pi}{3}, \frac{\pi}{4}\right\}$.

Exercise 12.2

Solve the following equation

1. (i) $\cos 2x = \frac{\sqrt{2}}{2}$

   (ii) $4\cos^2 x - 1 = 0$

2. (i) $4\sin^2\left(\frac{1}{2}x\right) = 1$

   (ii) $4\cos^2 x - 3 = 0$

3. (i) $\sin^2\left(\frac{x}{2}\right) - \frac{1}{2}\cos x = 0$

   (ii) $\tan x + \cot x = 4$

4. $4\cos^2\frac{x}{2} - 3 = 0$

5. $\cot x + 1 = \sqrt{3}(1 + \tan x)$

6. (i) $\sin x \cos x - \cos^2 x = 0$

   (ii) $\cos^2 x + \sin x = 1$

7. (i) $2\cos^2 x - \sin x = 1$

   (ii) $\sin 2x + \sin x = 0$

8. (i) $\sin 2x = \sin x$

   (ii) $\cos^2 2x + 3\sin 2x - 3 = 0$

9. (i) $4\cos^2 x + 2(\sqrt{3} - 1)\cos x - \sqrt{3} = 0$

   (ii) $4\cos^2 x - 4\cos x - 3 = 0$

10. (i) $\sin x + \cos x = 0$

(ii) $\sqrt{3}\cos\theta + \sin\theta = 1$

11. (i) $\sin 3\alpha - \cos 2\alpha - \sin\alpha = 0$

(ii) $\sin 3\theta = \cos 2\theta$.

12. (i) $\sec^2 x - \sqrt{3}\tan x = 1$

(ii) $\tan x + 2\sec x - 1 = 0$

13. (i) $\cos 2x + \cos x + 1 = 0$

(ii) $2\sin x = \operatorname{cosec} x + 1$

(iii) $\sqrt{2}\sec\theta + \tan\theta = 1$

(iv) $\operatorname{cosec} x = \sqrt{3} + \cot x$

(v) $\operatorname{cosec} x + \cot x = \sqrt{3}$

14. (i) $\sin 4x + \sin 2x = 0$

(ii) $\sin 3x + \sin 2x + \sin x = 0$

(iii) $\sin 5x - \sin 3x - \sin x = 0$

15. (i) $\cos 3x + \cos 2x + \cos x = 0$

(ii) $\cos\theta + \cos 3\theta = \cos 2\theta$

(iii) $\sin x + \cos 3x = \cos 5x$

16. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$

17. (i) $4\,\mathrm{Sin}^2 x = 1$

(ii) $2\sqrt{2}\,\mathrm{Cos}^2 x + (2 - \sqrt{2})\,\mathrm{Cos}\,x - 1 = 0$

(iii) $\mathrm{Cot}^2 x + (\sqrt{3} - 1)\,\mathrm{Cot}\,x - \sqrt{3} = 0$

ANSWERS

Exercise 1.1

1. (i) finite (ii) finite (iii) infinite
   (iv) finite (v) infinite (vi) finite
   (vii) finite
2. (i) true (ii) true (iii) false
   (iv) true
3. (i) 8 (ii) 32 (iii) 2^{18}
7. {2, 7, 13, 19}, {1, 6, 8, 14, 17}
8. (i) {3,-1} (ii) { }

Exercise 1.2

1.

	(i)	(ii)	(iii)	(iv)
Addition	Yes	No	No	No
Multiplication	Yes	Yes	No	Yes

2. (i) Commutative property of addition.
   (ii) Associative property of addition.
   (iii) Associative property of addition.
   (iv) Property of additive identity element.

(v) Property of multiplicative identity element.

(vi) Multiplication-addition property followed by property of additive inverses.

(vii) Property of additive inverses.

(viii) Commutative property of multiplication.

(ix) Multiplication - subtraction property (Distributivity of multiplication over subtraction).

(x) As for (ix).

(xi) Associativity of multiplication.

(xii) Distributivity of multiplication over addition and subtraction. (Multiplication - addition and multiplication - subtraction properties).

3. (i) Additive property

(ii) Multiplicative property

(iii) Additive property ($-u$ added to each side).

(iv) Multiplicative property (each side multiplied by -1).

(v) Multiplicative property (each side multiplied by $\frac{1}{ab}$.

(vi) Multiplicative property.

6. (i) $1 + 4x$ (ii) 9

(iii) $\dfrac{ab + bc}{ad - bc}$ (iv) $\dfrac{b - a}{ab - 1}$

Exercise 1.3

4. (i) i (ii) -1 (iii) $-i$

5. (i) bi (ii) $\sqrt{5}\, i$ (iii) $\frac{4}{5} i$

6. (10, 4) 7. (15, −9) 8. (−36, 32)

9. $(-23, 2)$ 10. $(-15, 0)$

11. $\left(\frac{24}{29}, \frac{2}{29}\right)$ 12. $\left(\frac{50}{73}, -\frac{36}{73}\right)$

14. (i) $\left(\frac{-4}{65}, \frac{-7}{65}\right)$ (ii) $\left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$

15. (i) $(a+2bi)(a-2bi)$ (ii) $(3a+4bi)(3a-4bi)$

(iii) $(\sqrt{2}\,x+\sqrt{3}\,yi)(\sqrt{2}\,x-\sqrt{3}\,yi)$

16. (i) $\frac{-27}{41}-\frac{38i}{41}$ (ii) $-6+\frac{5}{2}i$

Exercise 1.4

2. (i) $\frac{1}{3}i$ (ii) $\frac{1}{5}+\frac{2}{5}i$ (iii) $\frac{-3}{34}+\frac{5i}{34}$

3. (i) i (ii) a^4

(iii) i (iv) -1

5. (i) $23+2i$ (ii) $9+\sqrt{3}\,i$

(iii) $\frac{2\sqrt{5}}{13}-\frac{4\sqrt{2}}{13}i$ (iv) $\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{3}}i$

8. (i) 1 (ii) 1

(iii) 1 (iv) a^2-b^2+2abi

(v) $\frac{a^2-b^2}{(a^2+b^2)^2}-\frac{2ab}{(a^2+b^2)^2}i$ (vi) $a^3-3ab^2+(3a^2b-b^3)i$

(vii) $a^3-3ab^2-(3a^2b-b^3)i$ (viii) $\frac{-9}{2197}+\frac{46}{2197}i$

10. (i) $-6+i$ (ii) $-4i-3$ (iii) $(-3, -1)$

12. (i) $\sqrt{218}$ (ii) $5+12i$

Exercise 2.1

1. (i) $\begin{pmatrix} 3 & 1 & 9 \\ 3 & 9 & 6 \\ 2 & 9 & 8 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 & -2 & 3 \\ 6 & 1 & -2 \end{pmatrix}$

   (iii) $\begin{pmatrix} 9 & -1 & -1 \\ 2 & 3 & 6 \\ 3 & 3 & -7 \end{pmatrix}$ (iv) $\begin{pmatrix} 10 \\ 3 \\ -3 \end{pmatrix}$

   (v) $\begin{pmatrix} 0 & 3 & 5 \\ -2 & 0 & 0 \\ 3 & 1 & 8 \end{pmatrix}$

2. (i) $\begin{pmatrix} 6 & 3 \\ 3 & 9 \\ 6 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & -5 & 5 \\ -15 & 5 & -10 \end{pmatrix}$

   (iii) $[-1]$ (iv) $[0]$

   (v) $\begin{pmatrix} -5 & -1 \\ 8 & -1 \end{pmatrix}$ (vi) $\begin{pmatrix} -1 & 0 & -2 \\ 1 & 2 & 8 \\ 0 & 1 & 3 \end{pmatrix}$

   (vii) $\begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix}$ (viii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. (i) $\begin{pmatrix} -1 & 1 & -\frac{1}{3} \\ -1 & -1 & -1 \\ -\frac{4}{3} & \frac{2}{3} & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & -2 & -1 \\ \frac{3}{2} & 4 & -1 \end{pmatrix}$

   (iii) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 3 \\ 4 & 2 & -4 \end{pmatrix}$

8. (i) Symmetric (ii) Symmetric

   (iii) Symmetric (iv) Not symmetric

9. (i) Diagonal (ii) Scalar (iii) Unit

   (iv) Scalar (v) Unit

Exercise 2.2

1. (i) 2 (ii) -4 (iii) 0

(iv) -2 (v) 0 (vi) x^3

(vii) $abc - af^2 - h^2c + 2ghf - g^2b$

3. (i) -5 (ii) 0 (iii) 12

(iv) -3 (v) 151 (vi) 10

6. (i) $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{5}{6} \\ \frac{-2}{3} & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ (ii) $\begin{bmatrix} -\frac{1}{3} & 1 & -\frac{4}{3} \\ \frac{1}{6} & 0 & -\frac{1}{3} \\ \frac{1}{6} & 0 & \frac{2}{3} \end{bmatrix}$

(iii) $\begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{3} \\ 0 & -1 & 0 \end{pmatrix}$ (iv) It has no solution.

(v) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ **(iv)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

7. (i) Non-singular (ii) Singular (iii) Singular

8. (i) $\lambda = 2$ (ii) $\lambda = -112$

9. (i) $x = 1, y = 1, z = 1$ (ii) $x = 0, y = 2, z = 0$

(iii) $x = 1, y = 1, z = 0$ (iv) $x = 1, y = -2, z = 3$

(v) The solution does not exist.

10. (i) $x = 1, y = 1, z = 0$ (ii) $x = -2, y = 0, z = 2$

(iii) It has no solution.

(iv) $x = -\frac{1}{3}, y = -\frac{25}{24}, z = -\frac{15}{24}$

(v) $x = -1, y = \frac{1}{2}, z = 2.$

(vi) $x = 3, y = -1, z = -2$

Exercise 3.1

1. (i) True (ii) True (iii) True
(iv) False (v) True
2. (i) True (ii) False (iii) False
(iv) False (v) False
3. (i) yes (ii) identity element = 4
(iii) 1 and 3 are inverses of each other, 2 itself, 4 itself.
4. '0'
7. (i) yes (ii) yes (iii) yes
8. (i) yes (ii) yes (iii) yes
9. yes, yes

Exercise 3.2

2. Addition 3. yes, yes, – 1
7. (i) False (ii) False
10. No, No 11. Not a group.

Exercise 4.1

1. 1 2. – 3 3. 75
4. 82 5. $p = -4, q = 1$ 6. $a = -2, b = -1$
7. 26
10. (i) 80 (ii) 24.

Exercise 4.2

1. $-61\,,\,3x^2-7x+20$ 2. $0\,,\,2x^2-x+9$

3. $3\,,\,x^2-2x-1$ 4. $0\,,\,x^2+4x+13$

5. 29 6. 12 7. 0

8. 15 9. $3\,,\,\pm i$ 10. $-1\,,\,3\pm 2i$

11. $1, 2, -1, -2$ 12. $-1, -1, -1, 4$

13. -1 14. 8

Exercise 4.3

1. $\{-6\,,\,1\}$ 2. $\{1\,,\,2\}$ 3. $\left\{-\frac{1}{2}, \frac{3}{4}\right\}$

4. $\left\{-\frac{5}{2}, \frac{1}{2}\right\}$ 5. $\left\{\frac{1}{2}, 1\right\}$ 6. $\left\{-3\,, \frac{3}{2}\right\}$

7. $\left\{-\frac{5}{2}, \frac{2}{3}\right\}$ 8. $\left\{-2\,, \frac{3}{2}\right\}$ 9. $\left\{-\frac{5}{4}, \frac{3}{2}\right\}$

10. $\left\{-\frac{2}{3}, \frac{3}{5}\right\}$ 11. $\frac{3\pm 3\sqrt{7}\,i}{2}$ 12. $\left\{\frac{3}{2}, \frac{2}{3}\right\}$

13. $\left\{-\frac{3}{5}, 1\right\}$ 14. $\left\{-\frac{1}{2}, \frac{25}{2}\right\}$ 15. $\left\{-\frac{1}{3}, 2\right\}$

16 $\left\{-\frac{3}{4}, -\frac{2}{3}\right\}$ 17 $\left\{2\,, \frac{3}{7}\right\}$ 18. $\left\{-\frac{3}{5}, -4\right\}$

19. $\left\{-\frac{5}{2}, \frac{4}{3}\right\}$ 20. $\left\{2\,, -\frac{1}{3}\right\}$

Exercise 4.4

1. $\{4\,,\,16\}$ 2. $\left\{\frac{1\pm\sqrt{-15}}{4}, \frac{-1\pm 3\sqrt{-7}}{8}\right\}$

3. $\left\{\pm 3\,, \pm\frac{\sqrt{-2}}{2}\right\}$ 4. $\{-8\,,\,64\}$ 5. $\left\{\frac{1}{4}, 4\right\}$

6. $\{4, -6\}$ 7. $\left\{2, -\frac{29}{8}\right\}$ 8. $\{-1, 4\}$

9. $\left\{-\frac{8}{3}, -\frac{49}{24}\right\}$ 10. $\left\{\frac{39}{8}, \frac{67}{15}\right\}$ 11. $\left\{5, -\frac{5}{16}\right\}$

12. $\left\{1, \frac{1}{2}, -2, \frac{7}{2}\right\}$ 13. $\left\{3, -1, \frac{1}{2}, \frac{3}{2}\right\}$

14. $\left\{1, 2, \frac{3 \pm \sqrt{37}}{2}\right\}$ 15. $\left\{0, 5, \frac{5 \pm \sqrt{5}}{2}\right\}$

16. $\{0, 4\}$ 17. $\left\{1, -\frac{25}{3}\right\}$

18. $\left\{4, \frac{11}{4}\right\}$ 19. $\{-2 \pm \sqrt{3}\}$

20. $\left\{\frac{-3 \pm \sqrt{6}}{2}\right\}$ 21. $\left\{1, 2, \frac{3 \pm \sqrt{-37}}{2,}\right\}$

22. $\left\{2, -7, \frac{-5 \pm \sqrt{71}}{2}\right\}$ 23. $\left\{\omega, \omega^2, \frac{-3 \pm \sqrt{5}}{2}\right\}$

24. $\{-1, 5 \pm 2\sqrt{6}\}$ 25. $\left\{\frac{-1 \pm \sqrt{5}}{2}, \frac{7 \pm \sqrt{53}}{2}\right\}$

26. $\left\{\frac{-5 \pm 3\sqrt{-3}}{2}, \frac{-5 \pm 3\sqrt{5}}{2}\right\}$ 27. $\left\{\frac{13}{5}\right\}$

28. $\{0, .26\}$ 29. $\{1, .58\}$

30. $\left\{-1, 1, \frac{\sqrt{3} \pm 1}{\sqrt{2}}, \frac{-\sqrt{3} \pm 1}{\sqrt{2}}\right\}$

Exercise 4.5

1. $\{(-6, 3)\}$ 2. $\left\{(1, 2), \left(\frac{-1}{29}, \frac{68}{29}\right)\right\}$

3. $\left\{\left(5, \frac{12}{5}\right), (-4, -3)\right\}$

4. $\left\{\left(\frac{7 + \sqrt{13}}{2}, \frac{-7 + \sqrt{13}}{2}\right), \left(\frac{7 - \sqrt{13}}{2}, \frac{-7 - \sqrt{13}}{2}\right)\right\}$

5. $\left\{(1, 2), \left(\frac{11}{5}, \frac{6}{5}\right)\right\}$ 6. $\left\{(1, 2), \left(\frac{3}{2}, \frac{1}{2}\right)\right\}$

7. $\left\{(4, 3), \left(\frac{25}{8}, \frac{25}{6}\right)\right\}$ 8. $\left\{(5, 2), \left(\frac{7}{2}, \frac{7}{2}\right)\right\}$

9. $\left\{(a, b), \left(\frac{a-b}{2}, \frac{b-a}{2}\right)\right\}$ 10. $\{(0, 2), (12-4)\}$

11. $\{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$

12. $\{(\sqrt{5}, \pm 1), (-\sqrt{5}, \pm 1)\}$

13. $\{(3, 2), (3, -2), (-3, 2), (-3, -2)\}$

14. $\left\{\left(\pm\sqrt{\frac{5}{2}}, \pm\sqrt{\frac{7}{2}}\right)\right\}$ 15. $\{(2, 0), (-2, 0)\}$

16 $\left\{(-2, -1), \left(\frac{8}{5}, \frac{31}{5}\right)\right\}$ 17. $\left\{(4, 4), \left(\frac{4}{25}, \frac{-28}{25}\right)\right\}$

18. $\left\{(1, 3), \left(\frac{4}{13}, \frac{-6}{13}\right)\right\}$ 19. $\left\{\left(\frac{-4}{5}, \frac{-3}{5}\right)\right\}$

20. $\left\{(3, -4), \left(\frac{-9}{5}, \frac{-12}{5}\right)\right\}$

21 $\left\{(1, 2), (-1, -2), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{-3}{2}, \frac{-1}{2}\right)\right\}$

$\left\{\left(\frac{1}{2}, 1\right), \left(\frac{-1}{2}, -1\right), \left(-\sqrt{2}, \frac{\sqrt{2}}{3}\right), \left(\sqrt{2}, \frac{-\sqrt{2}}{3}\right)\right\}$

23. $\left\{(4, 3), (-4, -3), \left(\sqrt{3}, \frac{1}{\sqrt{3}}\right), \left(\sqrt{3}, -\frac{1}{\sqrt{3}}\right)\right\}$

24. $\{(1, 1), (-1, -1), (3, 2), (-3, -2)\}$

25. $\{(\sqrt{3}, -2\sqrt{3}), (-\sqrt{3}, 2\sqrt{3}), (2\sqrt{2}, \sqrt{2}), (-2\sqrt{2}, -\sqrt{2})\}$

26. $\{(6, -3)\}$ 27. $\{(3, 1), (-1, -3)\}$

28. $\{(2, 1), (1, 2)\}$

29. $\{3, 2), (2, 3), (-3, -2), (-2, -3)\}$

30. $\left\{\left(\frac{1}{2}, 1\right), \left(\frac{-1}{2}, 1\right), \left(\frac{1}{2}, -1\right), \left(\frac{-1}{2}, -1\right)\right\}$

Exercise 4.6

1. (i) real and rational (ii) equal

   (iii) imaginary (iv) real and irrational

   (v) imaginary

2. (i) $1, \frac{-11}{9}$ (ii) $2, \frac{-2}{9}$ (iii) $-1, \frac{-21}{17}$

5. ± 3 (10) 0, 3, 4, -1

Exercise 4.7

1. (i) $x^2+x-6=0$ (ii) $x^2+x=0$

   (iii) $x^2-x-12=0$ (iv) $x^2-6x+10=0$

   (v) $x^2-4x+1=0$

2. (i) 1, 1 (ii) $\frac{-5}{2}, \frac{-1}{2}$ (iii) $\frac{5}{7}, \frac{4}{7}$

   (iv) $\frac{-2}{3}, \frac{1}{3}$ (v) $0, -9$ (vi) $-\frac{1}{5}, \frac{-7}{5}$

3. (i) $\frac{12}{5}$ (ii) $-\frac{7}{18}$ (iii) $\frac{-9}{5}$

4. (i) $3x^2-5x-7=0$ (ii) $5x^2+6x+8=0$

   (iii) $4x^2-11x+12=0$

5. (i) $x^2-8x+8=0$ (ii) $x^2-12x+4=0$

   (iii) $x^2-40x+8=0$ (iv) $x^2-8x+14=0$

   (v) $2x^2-4x+1=0$ (vi) $2x^2-12x+17=0$

   (vii) $x^2-24x+128=0$

6. $p^4x^2-2p^2(q^2-2pr)x+q^2(q^2-4pr)=0$

7. (i) $pr(p+r)+q^3=3pqr$

(ii) $p^{\frac{1}{4}}\, r^{\frac{1}{4}}\left(p^{\frac{1}{2}} + r^{\frac{1}{2}}\right) + q = 0$

8. $k = -5$ 9. $x^2 - 4x + 13 = 0$

10. $x^2 - 6x + 7 = 0$ 13. $n\,q^2 = pr\,(1+n)^2$

Exercise 4.8

1. (i) $-1, -\omega, -\omega^2$ (ii) $-2, -2\,\omega, -2\,\omega^2$

(iii) $3, 3\,\omega, 3\,\omega^2$

3. (i) 32 (ii) -256.

Exercise 4.9

1 22 and 26 meters 2. $2\frac{1}{2}\%$

3. 25% 4. 2, -3 5. 16, 8 meters

6. 13, 14 7. 5, 6, 7 8. 81

9. (3, 5), (5, 3) 10. 7, -6 11. 35

12. 63 13. 18 years 14. 19, 21

15. 451 16. 12, 20 17. 8, 15; -15, -8

18. 8, 12 centimeters 19. 7, 12

20. 5, 12, 13 cms. 21. 10, 15 meters.

22. $3\frac{1}{13}$ days 23. 3, 2 and $\frac{4}{3}, \frac{11}{3}$

24. $-1, -1$ and $\frac{3}{5}, \frac{5}{3}$

25. 2, 4 26. 4, -7 27. 3, $\frac{-4}{3}$

28. 3, $\frac{2}{3}$ 29. -5, 9 30. $\frac{2}{3}, \frac{3}{2}$

Exercise 5.1

1. $-4, -3, -2, -1$
2. $\frac{-1}{2}, 1, \frac{7}{2}, 7$
3. $3, 2, \frac{5}{3}, \frac{3}{2}$
4. 0, 1, 3, 6
5. $1, -1, 1, -1$
6. $1, 0, \frac{-1}{3}, \frac{1}{2}$
7. 4, 7, 10, 13
8. $2, -6, 18, -54$
9. $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}$
10. 20, 24
11. 14, 17
12. $-1, 1$
13. x^6, x^7
14. $-x^9, x^{11}$

Exercise 5.2

1. 17, 21
2. 15, 19
3. $-13, -17$
4. $x+3, x+4$
5. $x+7a, x+9a$
6. $2x+10, 2x+13$
7. $4x, 5x$
8. 31
9. $7\frac{1}{2}$
10. 12
11. 41
12. -30
13. 28th
14. 20th
15. $d = 4$
16. $n = 9$
17. $a = -4$
18. $a_n = a_{20} = -46$
19. (i) 5, 9, 13, 17, .. (ii) 2, 5, 8, 11, ...

Exercise 5.3

1. 592
2. $\frac{35}{2}$
3. 36

4. 2 5. $\sqrt{5}$ 6. $\frac{191}{2}$

7. 10, 25 8. 9, 6, 3, 0, – 3, – 6

9. $2, \frac{7}{2}, 5, \frac{13}{2}$ 10. $\frac{-25}{2}, -7, \frac{-3}{2}$

11. 0

Exercise 5.4

1. $\frac{n}{2}(3n+7)$ 2. 354 3. 475

4. 120 5. $36p + 8q$ 6. $12l - 78m$

7. $\frac{170}{3}$ 8. – 40 9. $n(3n-4)$

10. 200000 11. $n(2n+3)$, 20300

12. 5, 8, 11 in either order.

13. 3, 5, 7, 9 14. 4, 5, 6, 7, 8

Exercise 5.5

1. 50, 250, 1250, 6250, 31250

2. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

3. $\frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$

4. $\frac{81}{2}$ 5. 1024

6. – 1536 7. $\frac{16}{625}$

8. 3, 15, 75, 375, 1875, 9375

9. 128, – 64, 32, – 16, 8, – 4, 2, – 1

10. 2, – 6, 18, – 54, 162

11. $81, 27, 9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

12. First term = 4, nth term = $4\left(\frac{1}{2}\right)^{n-1}$

13. $a = 3$ 14. $n = 6$ 15. $r = -\frac{1}{4}$

16. $l = \frac{1}{4}$ 17. 27635.63

18. 262351 19. 5.06 dm

Exercise 5.6

1. ± 16 2. ± 9 3. ± 1

4. 64, 16, 4 5. – 6, 18, – 54, 162

6. $\frac{1}{6}, \frac{1}{18}, \frac{1}{54}$ 7. $n = -\frac{1}{2}$

Exercise 5.7

1. $\frac{3}{2}(3^n - 1)$ 2. $5(2^n - 1)$ 3. $\frac{364}{243}$

4. $\frac{x(1 - x^{20})}{1 - x}$ 5. 341 6. $9\left(1 - \frac{1}{3^{10}}\right)$

7. $\frac{1}{3}\left(\frac{10^6 - 1}{10^6}\right)$ 8. $\frac{2^{10} - 1}{8}$

9. 26.76 dm (nearly). 10. 62

11. (i) $\frac{2}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$ (ii) $\frac{1}{3}\left[\frac{10}{9}(10^n - 1) - n\right]$

(iii) $\dfrac{1}{x-y}\left[\dfrac{x^2(x^n-1)}{x-1}-\dfrac{y^2(y^n-1)}{y-1}\right]$

(iv) $\dfrac{1}{1-x}\left[\dfrac{1-r^n}{1-r}-\dfrac{x(1-x^n r^n)}{1-xr}\right]$

12. Rs.402271.44

Exercise 5.8

1. divergent
2. convergent
3. convergent
4. convergent
5. divergent
6. divergent
7. 9
8. 12
9. $\frac{256}{3}$
10. 9
11. 4
12. $\frac{1}{4}$
13. $\frac{9}{11}$
14. $\frac{31}{99}$
15. $2\frac{410}{999}$
16. $3\frac{1}{37}$
17. $\frac{5}{6}$
18. $\frac{553}{1100}$

Exercise 5.9

1. $\frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{1}{4}, \frac{2}{9}$
2. $\frac{1}{13}, \frac{1}{16}, \frac{1}{19}$
3. $-\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, -\frac{1}{11}$
4. $\frac{1}{5}$
5. $-\frac{1}{17}$
6. $\frac{2}{39}$
7. $\frac{1}{7}$
8. $-\frac{1}{18}$
9. $-\frac{1}{26}$
10. $\frac{2}{11}$

Exercise 5.10

1. $\frac{1}{14}$ 2. $\frac{99}{10}$ 3. $\frac{12}{25}$

4. $\frac{-48}{7}$ 5. $1, \frac{5}{8}$

6. $\frac{35}{23}, \frac{35}{31}, \frac{35}{39}, \frac{35}{47}$

Exercise 5.11

1. Rs.23500

2. 66 balls in the bottom layer; 286 balls in all.

3. 7, 12, 17 4. Rs 717,000 5. 125 meters

6. $\frac{10}{9}$ 7. 50% 8. Rs 402028

9. 50% 10. 1, 2, 3 or 17, 2, – 13.

11. – 5, 1 and 7 or 7, 1 and – 5. 12. Rs 2552.56

Exercise 6.1

1. (i) 6840 (ii) 43680 (iii) 95040
   (iv) 604800 (v) 362880

2. (i) 6 (ii) 3 (iii) 9

4. 120 5. 1956 6. 5040

7. 120 8. 90

9. (i) 48 (ii) 72 10. 600, 120

11. 30240 12. 1440 13. 1440 14. 54

Exercise 6.2

1. (i) 15120 (ii) 20160 (iii) 4989600

(iv) 10810800 (v) 1663200

2. 360 3. 15 4. 60060

5. (a) 120 (b) 240

Exercise 6.3

1. (i) 9 (ii) 12

(iii) 18 (iv) 7

2. (i) $n = 12$, $r = 3$ (ii) $n = 7$, $r = 3$

(iii) $n = 10$, $r = 5$.

3. 35 4. 56 5. 1365, 1001

7. (i) 11760 (ii) 22968 (iii) 20616

Exercise 6.4

1		Outcomes of Sample space	Outcomes of an event	Probability
(a)	(i)	1, 2, 3, 4, 5, 6	2, 4, 6	1/2
	(ii)	1, 2, 3, 4, 5, 6	1, 3, 5	1/2
	(iii)	1, 2, 3, 4, 5, 6	1, 3, 5	1/2
	(iv)	1, 2, 3, 4, 5, 6	1, 2, 3, 4, 5, 6	1
(b)	(i)	HH, HT, TH, TT	HT, TH	1/2
	(ii)	HH, HT, TH, TT	HH, HT, TH	3/4
(c)	(i)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	2, 4, 6, 8, 10	1/2
	(ii)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	1, 3, 5, 7, 9	1/2
	(iii)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	1, 3, 5, 7, 9	1/2
	(iv)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	1, 2, 3, 4	2/5
	(v)	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	———	0

	(vi) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	1
(d)	(i) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10	1	1/10
	(ii) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10	1	1/10
	(iii) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10	(1, 1)	1/100
	(iv) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10	2-2, 4-2, 6-2, 8-2, 10-2	1/20

2. $\frac{5}{8}, \frac{3}{8}$ 3. $\frac{1}{3}$

Exercise 6.5

1. (a) 14 (b) 16 (c) 30
(d) 7/15 (e) 8/15
2. (a) 4 (b) 5 (c) 25
3. (a) 120 (b) 600 (c) 1200
(d) 12000
4. 0.12 (b) $\frac{3}{5}$

Exercise 6.6

1. 3/4 2. 0.2 3. 5/11
4. 20 5. 2/5 6. 20, 9/10
7. Rs.2.72

Exercise 6.7

1. 12 2. 2730
3. (a) 60 (b) 125

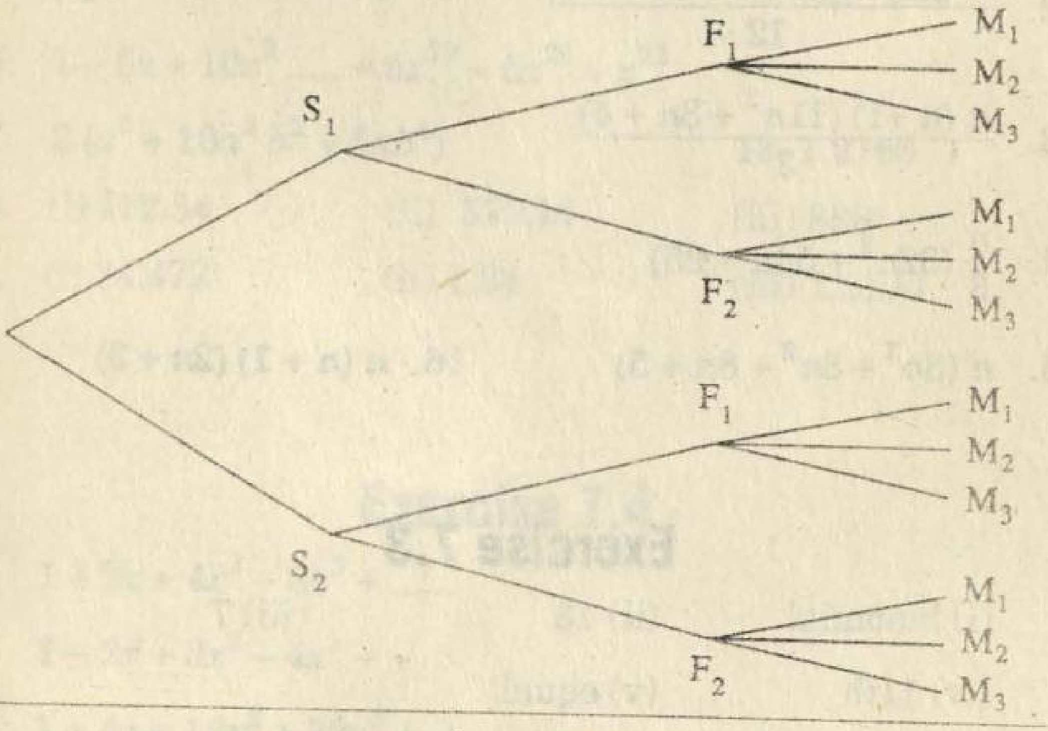
(c) 50 (d) 50

4. (a) 4/9 (b) 1/2 (c) 8/45

5. (a) 32 (b) 5/32

6. (a) 1/4 (b) 27

7. (a) 1/16 (b) 9/16 (c) 3/16



S_1
S_2
F_1
F_2
F_1
F_2
M_1
M_2
M_3
M_1
M_2
M_3
M_1
M_2
M_3
M_1
M_2
M_3

Tree diagram for Question 1

Two Soups : S_1, S_2

Two Fish : F_1, F_2

Three Meats : M_1, M_2, M_3

Exercise 7.2

1. $\frac{n(n+1)(n+2)}{3}$

2. $\frac{n}{6}(2n^2+21n+73)$

3. $\frac{2n(n+1)(2n+1)}{3}$

4. $\frac{n}{3}(4n^2-1)$

5 $\frac{n}{3}(4n^2+6n-1)$

6. $n(3n^2+3n-2)$

7. $n^2(2n^2-1)$ 8. $n(2n^3+8n^2+7n-2)$

9. $\frac{n(n+1)(3n+1)(n+2)}{12}$

10. $n(16n^3-16n^2-2n+3)$

11 $\frac{n}{3}(16n^2-12n-1)$

12. $\frac{n(n+1)(3n^2+19n+32)}{12}$

13. $\frac{n(n+1)(11n^2+3n+4)}{3}$

14. $\frac{n}{3}(32n^2+54n+25)$

15. $n(8n^3+8n^2+8n+5)$ 16. $n(n+1)(2n+3)$

Exercise 7.3

1. (i) binomial (ii) 13 (iii) 7

(iv) 11th (v) equal

2. (i) $64x^6+192x^5y+240x^4y^2+160x^3y^3+60x^2y^4+12xy^5+y^6$

(ii) $x^7+7x^5+21x^3+35x+\frac{35}{x}+\frac{21}{x^3}+\frac{7}{x^5}+\frac{1}{x^7}$

(iii) $256x^8+1024x^6+1792x^4+1792x^2+1120+\frac{448}{x^2}+\frac{112}{x^4}+\frac{16}{x^6}+\frac{1}{x^8}$

(iv) $64x^{12}-64x^9+\frac{80}{3}x^6-\frac{160}{27}x^3+\frac{20}{27}-\frac{4}{81x^3}+\frac{1}{729x^6}$

(v) $\frac{x^4}{16}-\frac{x^3}{y}+6\frac{x^2}{y^2}-16\frac{x^3}{y^3}+\frac{16}{y^4}$

3. 1913.625

4. $\frac{(2n)!}{(n-1)!\,(n+1)!}(-1)^{n-1}x^{n-1}$

5. $101376\left(\frac{1}{x}\right)$ 6. $1.09375\,x^{11}$ 7. 113.75

8. -33.1852 9. $T_7 = 672$ 10. $T_3 = 5$

11. $T_5 = 126\,x^5y^4$ 12. $T_9 = 495x^4\,y^4$ 13. $T_4 = 56x^{5/2}\,y^6$

14. $T_6 = -193.375\,x^{24}\,y^{15}$ 15. $T_4 = 35x^9$

16. $1 - 5x + 10x^2 \ldots\ldots + 5x^{20} - 5x^{21} + x^{25}$

17. $2\,(a^5 + 10a^3\,b^2 + 5ab^4)$ 18. 1.2166

19. (i) 112.54 (ii) 570.14 (iii) 888

21. (i) 34.472 (ii) 1.34 (iii) 1.2190

Exercise 7.4

1. $1 + 2x + 4x^2 + 8x^3 + \ldots\ldots$

2. $1 - 2x + 3x^2 - 4x^3 + \ldots\ldots$

3. $1 + 4x + 12x^2 + 32x^3 + \ldots\ldots$

4. $2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \ldots\ldots$

5. $5^{1/3}\left(1 - \frac{x}{15} - \frac{x^2}{225} - \frac{x^3}{2025} + \ldots\ldots\right)$

6. $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$

7. $1 + 3x + 6x^2 + 10x^3 + \ldots\ldots$

8. $1 + \frac{x}{2} + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots\ldots$

9. $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \ldots\ldots$

10. 4.796 11. 3.0723 12. 2.8481

13. 7.211 14. 2.0123 15. 3.458

16. 8.927

18. (i) $\sqrt{\frac{2}{3}}$ (ii) $\sqrt{3}$ (iii) $\left(\frac{1}{2}\right)^{\frac{3}{4}}$

22. 2% 23. 3%

Exercise 8.1

1. $\frac{2\pi}{3} = 2.09$ 2. $\frac{7\pi}{6} = 3.66$ 3. $\frac{7\pi}{4} = 5.50$

4. $3\pi = 9.42$ 5. $\frac{1}{9} = 0.11$ 6. $\frac{\pi}{12} = \cdot 26$

7. $\frac{\pi}{8} = .39$ 8. 0.65 9. 0.74

10. 225° 11. 120° 12. 540°

13. $316^\circ.27$ 14. $18^\circ.24$ 15. $24^\circ.32$

16. $258^\circ.98$ 17. $157^\circ.56$ 18. 315°

19. (i) 2π (ii) $\frac{\pi}{6}$ 20. 2 radians

21. (a) 2 radians (b) 3.14 radians

22. (a) 6 cm (b) 16.5 cm (c) $\frac{5\pi}{2}$ *cm*

23. 3 cm, 5.5 cm 24. 6700 km (nearly)

26. 1 sq m 27. 14.14 cm 28. 3354 km

29. 1.39×10^6 km

Exercise 8.2

1. (i) Along $\overrightarrow{OY'}$ (ii) Along $\overrightarrow{OX'}$ (iii) Along $\overrightarrow{OX'}$

(iv) Along $\overrightarrow{OX'}$ (v) Along $\overrightarrow{OY}$ (vi) In Quadrant II

(vii) In Quad. I (viii) In Quad. III (ix) Along $\overrightarrow{OX}$
(x) Along $\overrightarrow{OX}$ (xi) Along $\overrightarrow{OX}$ (xii) Along $\overrightarrow{OX}$

2. (i) + ve (ii) – ve (iii) + ve
(iv) – ve (v) – ve (vi) + ve
(vii) – ve (viii) + ve (ix) – ve
(x) + ve (xi) – ve (xii) + ve
(xiii) – ve (xiv) + ve (xv) + ve

24. $\sin\theta = \sqrt{1-\cos^2\theta}$, $\tan\theta = \dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$,

$$sec\ \theta = \frac{1}{\cos\theta},\ cosec\ \theta = \frac{1}{\sqrt{1-\cos^2\theta}}$$

25. – 3/5, – 4/3, – 3/4, – 5/3, 5/4

26. – 3/5, – 4/5, 3/4, – 5/4, – 5/3

27. $-\dfrac{\sqrt{5}}{3}, -\dfrac{2}{\sqrt{5}}, -\dfrac{\sqrt{5}}{2}, -\dfrac{3}{\sqrt{5}}, \dfrac{3}{2}$

28. $\dfrac{1}{\sqrt{10}}, -\dfrac{3}{\sqrt{10}}, -3, -\dfrac{\sqrt{10}}{3}, \sqrt{10}$

29. $-\dfrac{1}{2}, \dfrac{1}{\sqrt{3}}, \sqrt{3}, -\dfrac{2}{\sqrt{3}}, -2$

30. (i) IV (ii) I (iii) III
(iv) IV (v) III (vi) II

Exercise 8.3

1. (i) 1 (ii) 0 (iii) $1/\sqrt{3}$

3. $\sin(-\pi) = 0$, $\cos(-\pi) = -1$ etc.

4. $\sin(-3\pi) = 0$, $\cos(-3\pi) = -1$ etc.

5. $\sin(-5\pi/2) = -1$, $\cos(-5\pi/2) = 0$ etc.

6. $\sin 9 - 9\pi/2) = -1$, $\cos(-9\pi/2) = 0$ etc.

7. $\sin(-15\pi) = 0$, $\cos(-15\pi) = -1$

8. $\sin(-1530^0) = -1$, $\cos(-1530^o) = 0$

9. $\sin(-2430^o) = 1$, $\cos(-2430^o) = 0$

10. $\sin\frac{235}{2}\pi = -1$, $\cos\frac{235}{2}\pi = 0$

11. $\sin\frac{407}{2}\pi = -1$, $\cos\left(\frac{407}{2}\pi\right) = 0$

Exercise 9.1

6. (i) $\frac{33}{65}$ (ii) $-\frac{56}{65}$

9. (i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ (iii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

10. (i) $\frac{-56}{65}$ (ii) $\frac{16}{65}$ (iii) $-\frac{33}{65}$ (iv) $\frac{63}{65}$

11. (a) (i) $\frac{24}{25}$ (ii) $\frac{7}{25}$ (iii) $\frac{24}{7}$

(b) (i) $-\frac{5}{13}$ (ii) $\frac{12}{13}$ (iii) $\frac{5}{12}$

16. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $\frac{1-\sqrt{3}}{2\sqrt{2}}$ (iii) $\frac{\sqrt{3}+1}{1-\sqrt{3}}$

Exercise 9.2

1. (i) $\frac{24}{7}$ (ii) $\frac{1}{\sqrt{10}}$ (iii) $\frac{3}{\sqrt{10}}$ (iv) $\frac{1}{3}$

2. (i) $\frac{-120}{169}$ (ii) $\frac{3}{\sqrt{13}}$ (iii) $\frac{2}{\sqrt{13}}$ (iv) $\frac{3}{2}$

3. (i) $\sqrt{\frac{9}{10}}$ (ii) $\sqrt{\frac{1}{10}}$

4. $\sqrt{\frac{3}{2}}$ 5. $\frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$

6. (i) $-\frac{1}{2}$ (ii) $\frac{\sqrt{3}}{2}$

Exercise 9.3

1. (i) $\cos 80^o + \cos 50^o$ (ii) $\sin 60^o - \sin 10^o$

(iii) $\sin 119^o - \sin 27^o$ (iv) $\sin 4A + \sin 2A$

(v) $\frac{1}{2}\left[\sin 62^o - \sin 6^o\right]$ (vi) $\frac{1}{2}\left[\cos 90^o + \cos 30^o\right]$

(vii) $\frac{1}{2}[\sin 8\theta - \sin 2\theta]$ (viii) $\frac{1}{2}[\sin 12\theta - \sin 2\theta]$

(ix) $\frac{1}{2}[\sin\alpha + \sin\beta]$ (x) $\frac{1}{2}[\cos A + \cos B]$

2. (i) $2\cos 39^o \cos 17^o$ (ii) $2\cos 80^o \sin 4^o$

(iii) $-2\sin 80^o \sin 4^o$ (iv) $2\sin 6\theta \cos 2\theta$

(v) $2\cos\frac{5\alpha}{4}\sin\frac{5\alpha}{12}$ (vi) $2\cos\frac{25\alpha}{24}\cos\frac{7\alpha}{24}$

(vii) $2\cos\alpha\cos\beta$ (viii) $2\sin\alpha\cos\beta$.

Exercise 10.1

	Domain	Period
1.	R	2π
2.	$R - \left\{x \mid x = (2k+1)\frac{\pi}{2},\ k \in Z\right\}$	π
3.	$R - \{x \mid x = k\pi,\ k \in Z\}$	2π
4.	R	2π
5.	R	$\frac{\pi}{3}$

6.	$R - \left\{x \mid x = (2k+1)\frac{3\pi}{8}, K \in Z\right\}$	$\frac{3}{4}\pi$
7.	$R - \{x \mid x = k\pi, k \in Z\}$	π
8.	$R - \{x \mid x = 4k\pi, k \in Z\}$	8π
9.	R	$\frac{\pi}{2}$

10. $\{y \mid -1 \le y \le 1, y \in R\}$ 11. $\{y \mid -5 \le y \le 5, y \in R\}$

12. R 13. $\{y \mid -1 \le y \le 1, y \in R\}$

14. $\{y \mid -1 \le y \le 1, y \in R\}$

15. R 16. R

17. $R - \{y \mid -1 < y < 1, y \in R\}$

18. $R - \{y \mid -1 < y < 1, y \in R\}$

Exercise 11.1

1. $b = 273.63$, $c = 370.64$, $\beta = 47^\circ 35'$
2. $b = 603.41$, $c = 742.11$, $\beta = 54^\circ 24'$
3. $a = 271.58$, $c = 358.49$, $\alpha = 49^o 15'$
4. $a = 366.09$, $c = 649.09$, $\alpha = 34^o 20'$
5. $a = 184.72$, $b = 140.37$ $\beta = 37^0 14'$
6. $a = 282.04$, $b = 297.58$, $\alpha = 43^o 28'$
7. $c = 563.98$, $\alpha = 52^o 56'$, $\beta = 37^o 04'$
8. $b = 383.62$, $\alpha = 44^o 44'$, $\beta = 45^o 16'$
9. $a = 584.66$, $\alpha = 67^o 41'$, $\beta = 22^o 19'$
10. $c = 6.83$, $\alpha = 38^o 28'$, $\beta = 51^o 32'$
11. $a = 136.96$, $c = 340.74$, $\beta = 66^o 18'$
12. $c = 280.54$, $b = 183.75$, $\alpha = 49^o 05'$

Exercise 11.2

1. 115.48 meters 2. 58°

3. 175.62 meters 4. 57° 31

5. **25.49** meters 6. 136.59 meters

7. 154.81 meters 8. 259.81 meters

9. 881.49 meters 10. 154.04 meters

11. 95.79 meters 12. 13444.5 meters

13. 120.93 meters 14. **403.3** meters

15. **33^{0} 42′** 16. 950.2, 728 meters

17. 80.37 meters 18. 189.29 meters

19. 30 meters **20. 92.46** meters

Exercise 11.3

1. 1.5 2. 35° 57′ **58″** 3. 0.4269

4. 0.7321 5. 0.8485 6. 31.06

7. **245.45** 8. 12.68 **9. 449.22**

10. **78.09** 11. 1654.46 m , 1578.70 m

12. **35.64** km.

Exercise 11.4

1. 13.31 2. 85.83 3. 34.82

4. 132.69 5. 7.21 6. 83.24

7. 49° 40′ 47″ , 35° 54′ 30″ 8. 33.59

9. 34.39 10. 10.28 11. 13.08

12. 35.18 13. 52° 37′ 11″ 14. 3962.48 m

Exercise 11.5

1. $a = 5.62$ $b = 5.54$ $\beta = 68^{\circ}$
2. $b = 554$ $c = 221.29$ $\alpha = 55^{\circ}$
3. $a = 70.78$ $c = 98.97$ $\alpha = 44^{\circ}\,27'$
4. $a = 79.82$ $c = 67.37$ $\gamma = 56^{\circ}\,05'$
5. $a = 34.20$ $c = 31.44$ $\gamma = 54^{\circ}\,27'$
6. $c = 40.68$ $\alpha = 81^{\circ}\,43'$ $\beta = 41^{\circ}\,17'$
7. $a = 15.31$ $\beta = 30^{\circ}\,26'$ $\gamma = 111^{\circ}\,14'$
8. $a = 14.74$ $\beta = 70^{\circ}\,39'$ $\gamma = 85^{\circ}\,56'$
9. $c = 89.34$ $\alpha = 72^{\circ}\,46'$ $\beta = 31^{\circ}\,23'$
10. $b = 408.98$ $\alpha = 22^{\circ}\,39'$ $\gamma = 46^{\circ}\,59'$
11. $\beta = 81^{0}\,56'$
12. $\beta = 79^{\circ}\,48'$
13. $\alpha = 96^{0}\,36'$ $\beta = 44^{0}\,16'$ $\gamma = 39^{0}\,8'$
14. $\alpha = 49^{\circ}\,59'$ $\beta = 80^{0}1'$ $\gamma = 50^{\circ}$
15. $\alpha = 58^{0}\,22'$ $\beta = 33^{\circ}\,44'$ $\gamma = 87^{0}\,55'$
16. $\alpha = 75^{\circ}\,43'$ $\beta = 43^{\circ}\,48'$ $\gamma = 60^{\circ}\,29'$
17. $\alpha = 52^{\circ}\,02'$ $\beta = 27^{\circ}\,59'$ $\gamma = 99^{\circ}\,59'$
18. $\alpha = 84^{\circ}\,20'$ $\beta = 31^{\circ}\,02'$ $\gamma = 64^{\circ}\,38'$
19. $\alpha = 31^{\circ}\,16'$ $\beta = 48^{\circ}\,52'$ $\gamma = 99^{\circ}\,52'$
20. $\alpha = 52^{\circ}\,10'$ $\beta = 32^{\circ}\,32'$ $\gamma = 95^{\circ}\,18'$
21. $\alpha = 27^{\circ}\,56'$ $\beta = 52^{\circ}\,46'$ $\gamma = 99^{\circ}\,18'$
22. $\alpha = 57^{\circ}\,18'$ $\beta = 46^{\circ}\,48'$ $\gamma = 75^{\circ}\,54'$
23. $\alpha = 43^{\circ}\,54'$ $\beta = 56^{\circ}\,18'$ $\gamma = 79^{\circ}\,48'$

24. $\alpha = 77^\circ\, 56'$ $\quad\beta = 40^\circ\, 48'$ $\quad\gamma = 61^\circ\, 16'$

Exercise 11.6

1. 426.66 2. 63195.70 3. 76661.62

4. 35705.89 5. 33482.01 6. 3.62

7. 400 8. 651.74 9. 614.46

10. 413.84 11. 804.13 12. 419.91

13. 134436.22 14. 2730.70 15. 5.66

Exercise 11.7

1. 41.48 , 62.22 , 51.49 2. 39.29 , 8.16 meters

21. 319.97 square decimeter 22. 33.07 dm.

23. 3676.68 sq. meters.

Exercise 12.1

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\sqrt{3}}{2}$

(iii) $\left\{2n\pi + \frac{\pi}{3}\right\} \cup \left\{2n\pi - \frac{\pi}{3}\right\}$, n $\in$ Z.

2. (i) $-\frac{\pi}{3}$ (ii) $\frac{3\pi}{4}$ (iii) $-\frac{\pi}{4}$

(iv) $\left\{2n\pi + \frac{5\pi}{4}\right\} \cup \left\{2n\pi - \frac{\pi}{4}\right\}$, n $\in$ Z.

(v) $\left\{n\pi + \frac{5\pi}{6}\right\}$, n $\in$ Z.

3. (i) $\frac{4}{5}$ (ii) $-\frac{3}{5}$ (iii) $\frac{5}{13}$

(iv) $-\sqrt{1-x^2}$

4. $\alpha = \beta$

8. (i) $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$ (ii) $\frac{1}{2}$ (iii) $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$

(iv) $\{2n\pi + \pi\}\ n \in Z$ (v) $-\frac{\pi}{4}$

(vi) $\approx -\frac{529\pi}{2700}$ (vii) $\frac{2\pi}{3}$ (viii) $-\frac{3}{4}$

(ix) not defined. (x) $\frac{4}{5}$ (xi) $-\frac{12}{3}$

(xii) $\frac{56}{65}$ (xiii) $-\frac{1}{8}$ (ivx) $\frac{1}{\sqrt{2}}$

11. $Tan^{-1}\left(\frac{5}{6}\right)$

Exercise 12.2

1. (i) $\left\{n\pi + \frac{\pi}{8}\right\} \cup \left\{n\pi + \frac{7\pi}{8}\right\}, n \in Z$

(ii) $\left\{2n\pi + \frac{\pi}{3}\right\} \cup \left\{2n\pi + \frac{5\pi}{3}\right\} \cup \left\{2n\pi + \frac{2\pi}{3}\right\} \cup \left\{2n\pi + \frac{4\pi}{3}\right\}, n \in Z$

2. (i) $\left\{4n\pi + \frac{\pi}{3}\right\} \cup \left\{4n\pi + \frac{5\pi}{3}\right\}, n \in Z$

(ii) $\left\{2n\pi - \frac{\pi}{6}\right\} \cup \left\{2n\pi + \frac{\pi}{6}\right\} \cup \left\{2n\pi + \frac{5\pi}{6}\right\} \cup \left\{2n\pi + \frac{7\pi}{6}\right\}, n \in Z$

3. (i) $\left\{4n\pi + \frac{\pi}{3}\right\} \cup \left\{4n\pi + \frac{5\pi}{3}\right\} \cup \left\{4n\pi + \frac{7\pi}{3}\right\} \cup \left\{4n\pi + \frac{11\pi}{3}\right\}, n \in Z$

(ii) $\left\{2\pi n - \frac{\pi}{12}\right\} \cup \left\{2\pi n + \frac{5\pi}{12}\right\} \cup \left\{n\pi + \frac{7\pi}{12}\right\} \cup \left\{2n\pi + \frac{\pi}{12}\right\}$

4. $\left\{4n\pi + \frac{\pi}{3}\right\} \cup \left\{4n\pi - \frac{\pi}{3}\right\} \cup \left\{4n\pi + \frac{5\pi}{3}\right\} \cup \left\{4n\pi + \frac{7\pi}{3}\right\}, n \in Z$

5. $\left\{n\pi + \frac{\pi}{6}\right\} \cup \left\{n\pi + \frac{3\pi}{4}\right\}, n \in Z$

6. (i) $\left\{n\pi + \frac{\pi}{4}\right\} \cup \left\{2n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi + \frac{3\pi}{2}\right\}, n \in Z$

(ii) $\{n\pi\} \cup \left\{2n\pi + \frac{\pi}{2}\right\}, n \in Z$

7. (i) $\left\{2n\pi + \frac{\pi}{6}\right\} \cup \left\{2n\pi + \frac{5\pi}{6}\right\} \cup \left\{2n\pi + \frac{3\pi}{2}\right\}, n \in Z$

(ii) $\{n\pi\} \cup \left\{2n\pi + \frac{2\pi}{3}\right\} \cup \left\{2n\pi + \frac{4\pi}{3}\right\}, n \in Z$

8. (i) $\{n\pi\} \cup \left\{2n\pi + \frac{\pi}{3}\right\} \cup \left\{2n\pi - \frac{\pi}{3}\right\}, n \in Z$

(ii) $\left\{n\pi + \frac{\pi}{4}\right\}, n \in Z$

9. (i) $\left\{2n\pi + \frac{\pi}{3}\right\} \cup \left\{2n\pi + \frac{5\pi}{3}\right\} \cup \left\{2n\pi + \frac{5\pi}{6}\right\} \cup \left\{2n\pi + \frac{7\pi}{6}\right\}, n \in Z$

(ii) $\left\{2n\pi + \frac{2\pi}{3}\right\} \cup \left\{2n\pi + \frac{4\pi}{3}\right\}, n \in Z$

10. (i) $\left\{2n\pi + \frac{3\pi}{4}\right\} \cup \left\{2n\pi + \frac{7\pi}{4}\right\}, n \in Z$

(ii) $\left\{2n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi - \frac{\pi}{6}\right\} \cup \left\{2n\pi + \frac{7\pi}{6}\right\}, n \in Z$

11. (i) $\left\{n\pi + \frac{\pi}{4}\right\} \cup \left\{n\pi + \frac{3\pi}{4}\right\} \cup \left\{2n\pi + \frac{\pi}{6}\right\} \cup \left\{2n\pi + \frac{5\pi}{6}\right\}, n \in Z$

(ii) $\left\{2n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi + \frac{\pi}{10}\right\} \cup \left\{2n\pi + \frac{9\pi}{10}\right\} \cup \left\{2n\pi + \frac{13\pi}{10}\right\} \cup \left\{2n\pi + \frac{17\pi}{10}\right\}, n \in Z$

12. (i) $\{n\pi\} \cup \left\{n\pi + \frac{\pi}{3}\right\}, n \in Z$

(ii) Solution does not exist.

13. (i) $\left\{n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi + \frac{2\pi}{3}\right\} \cup \left\{2n\pi + \frac{4\pi}{3}\right\}, n \in Z$

(ii) $\left\{2n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi + \frac{7\pi}{6}\right\} \cup \left\{2n\pi - \frac{\pi}{6}\right\}, n \in Z$

(iii) $\left\{(2n + 1)\pi + \frac{3\pi}{4}\right\}, n \in Z$

(iv) $\left\{ n\pi + \frac{2\pi}{3} \right\}, n \in Z$

(v) $\left\{ n\pi + \frac{\pi}{3} \right\}, n \in Z$

14. (i) $\left\{ \frac{n\pi}{3} \right\} \cup \left\{ n\pi + \frac{\pi}{2} \right\}, n \in Z$

(ii) $\left\{ \frac{n\pi}{2} \right\} \cup \left\{ 2n\pi + \frac{2\pi}{3} \right\} \cup \left\{ 2n\pi + \frac{4\pi}{3} \right\}, n \in Z$

(iii) $\{n\pi\} \cup \left\{ \frac{n\pi}{2} + \frac{\pi}{12} \right\} \cup \left\{ \frac{n\pi}{2} + \frac{5\pi}{12} \right\}, n \in Z$

15. (i) $\left\{ \frac{n\pi}{2} + \frac{\pi}{4} \right\} \cup \left\{ 2n\pi + \frac{2\pi}{3} \right\} \cup \left\{ 2n\pi + \frac{4\pi}{3} \right\}, n \in Z$

(ii) $\left\{ \frac{n\pi}{2} + \frac{\pi}{4} \right\} \cup \left\{ 2n\pi + \frac{\pi}{3} \right\} \cup \left\{ 2n\pi + \frac{5\pi}{3} \right\}, n \in Z$

(iii) $\{n\pi\} \cup \left\{ \frac{n\pi}{2} + \frac{7\pi}{24} \right\} \cup \left\{ \frac{n\pi}{2} + \frac{11\pi}{24} \right\}, n \in Z$

16. $\left\{ n\pi + \frac{\pi}{2} \right\} \cup \left\{ \frac{n\pi}{2} + \frac{\pi}{4} \right\} \cup \left\{ \frac{n\pi}{4} + \frac{\pi}{8} \right\}, n \in Z$

17. (i) $\left\{ -\frac{\pi}{6}, \frac{\pi}{6} \right\}$ (ii) $\left\{ \frac{\pi}{3}, \frac{3\pi}{4} \right\}$ (iii) $\left\{ -\frac{\pi}{6}, \frac{\pi}{4} \right\}$

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	24
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	4 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

ANTI-LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2734	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2318	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 4 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 4 4	5 6 6

ANTI-LOGARITHMS

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·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	10	18	20

NATURAL SINES

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences				
													1′	2′	3′	4′	5′
0	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89	3	6	9	12	15
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	3	6	9	12	15
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	0523	87	3	6	9	12	15
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	0698	86	3	6	9	12	15
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	0872	85	3	6	9	12	15
5	0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	1045	84	3	6	9	12	14
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	1219	83	3	6	9	12	14
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	1392	82	3	6	9	12	14
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	1564	81	3	6	9	12	14
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	1736	80	3	6	9	12	14
10	1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	1908	79	3	6	9	12	14
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2079	78	3	6	9	11	14
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	2250	77	3	6	9	11	14
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2419	76	3	6	8	11	14
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	2588	75	3	6	8	11	14
15	2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2756	74	3	6	8	11	14
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2924	73	3	6	8	11	14
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3090	72	3	6	8	11	14
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3256	71	3	6	8	11	14
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3420	70	3	5	8	11	14
20	3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3584	69	3	5	8	11	14
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3746	68	3	5	8	11	14
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3907	67	3	5	8	11	14
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	4067	66	3	5	8	11	14
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	4226	65	3	5	8	11	13
25	4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	4384	64	3	5	8	11	13
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	4540	63	3	5	8	10	13
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	4695	62	3	5	8	10	13
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	4848	61	3	5	8	10	13
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	5000	60	3	5	8	10	13
30	5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	5150	59	3	5	8	10	13
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	5299	58	2	5	7	10	12
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	5446	57	2	5	7	10	12
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	5592	56	2	5	7	10	12
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	5736	55	2	5	7	10	12
35	5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	5878	54	2	5	7	10	12
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	6018	53	2	5	7	9	12
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	6157	52	2	5	7	9	12
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	6293	51	2	5	7	9	11
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	6428	50	2	4	7	9	11
40	6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	6561	49	2	4	7	9	11
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	6691	48	2	4	7	9	11
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	6820	47	2	4	6	9	11
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	6947	46	2	4	6	8	11
44	6947	6959	6972	6984	6997	7009	7022	7034	7045	7059	7071	45	2	4	6	8	10
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

NATURAL COSINES

NATURAL SINES

Degrees	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	60'	Degrees	Mean Differences				
													1'	2'	3'	4'	5'
45	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	·7193	44	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	·7314	43	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	·7431	42	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	·7547	41	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	·7660	40	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	·7771	39	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	·7880	38	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	·7986	37	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	·8090	36	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	·8192	35	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	·8290	34	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	·8387	33	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	·8480	32	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	·8572	31	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	·8660	30	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	·8746	29	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	·8829	28	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	·8910	27	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	·8988	26	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	·9063	25	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	·9135	24	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	·9205	23	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	·9272	22	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	·9336	21	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	·9397	20	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	·9455	19	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	·9511	18	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	·9563	17	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	·9613	16	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	·9659	15	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	·9703	14	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	·9744	13	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	·9781	12	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	·9816	11	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	·9848	10	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	·9877	9	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	·9903	8	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	·9925	7	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	·9945	6	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	·9962	5	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	·9976	4	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	·9986	3	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	·9994	2	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	·9998	1	0	0	0	0	0
89	·9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0	0
	60'	54'	48'	42'	36'	30'	24'	18'	12'	6'	0'		1'	2'	3'	4'	5'

NATURAL COSINES

NATURAL TANGENTS

Degrees	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	60'	Degrees	Mean Differences				
													1'	2'	3'	4'	5'
0	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89	3	6	9	12	15
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	3	6	9	12	15
2	0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	0524	87	3	6	9	12	15
3	0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	0699	86	3	6	9	12	15
4	0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	0875	85	3	6	9	12	15
5	0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	1051	84	3	6	9	12	15
6	1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	1228	83	3	6	9	12	15
7	1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	1405	82	3	6	9	12	15
8	1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	1584	81	3	6	9	12	15
9	1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	1763	80	3	6	9	12	15
10	1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	1944	79	3	6	9	12	15
11	1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	2126	78	3	6	9	12	15
12	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	2309	77	3	6	9	12	15
13	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	2493	76	3	6	9	12	15
14	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	2679	75	3	6	9	12	16
15	2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	2867	74	3	6	9	13	16
16	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3057	73	3	6	9	13	16
17	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3249	72	3	6	10	13	16
18	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3443	71	3	6	10	13	16
19	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3640	70	3	7	10	13	16
20	3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3839	69	3	7	10	13	17
21	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4040	68	3	7	10	13	17
22	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4245	67	3	7	10	14	17
23	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4452	66	3	7	10	14	17
24	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4663	65	4	7	11	14	18
25	4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4877	64	4	7	11	14	18
26	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	5095	63	4	7	11	15	18
27	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	5317	62	4	7	11	15	18
28	5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5543	61	4	8	11	15	19
29	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	5774	60	4	8	12	15	19
30	5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	6009	59	4	8	12	16	20
31	6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	6249	58	4	8	12	16	20
32	6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	6494	57	4	8	12	16	20
33	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	6745	56	4	8	13	17	21
34	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	7002	55	4	9	13	17	21
35	7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	7265	54	4	9	13	18	22
36	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	7536	53	5	9	14	18	23
37	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	7813	52	5	9	14	18	23
38	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	8098	51	5	9	14	19	24
39	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	8391	50	5	10	15	20	24
40	8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	8693	49	5	10	15	20	25
41	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	9004	48	5	10	16	21	26
42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	47	5	11	16	21	27
43	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	9657	46	6	11	17	22	28
44	9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	0000	45	6	11	17	23	29
	60'	54'	48'	42'	36'	30'	24'	18'	12'	6'	0'		1'	2'	3'	4'	5'

NATURAL COTANGENTS

NATURAL TANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences 1′	2′	3′	4′	5′
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	0355	44	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	0724	43	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	1106	42	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	1504	41	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	1918	40	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	2349	39	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	2799	38	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	3270	37	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	3764	36	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	4281	35	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	4826	34	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	5399	33	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	6003	32	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	6643	31	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	7321	30	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	966	8040	29	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	8807	28	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	9626	27	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	2·0057	2·0145	2·0233	2·0323	2·0413	2·0503	26	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	1445	25	16	31	47	63	78
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	2460	24	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	3559	23	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	4751	22	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	6051	21	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	7475	20	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	9042	19	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	3·0061	3·0237	3·0415	3·0595	3·0777	18	29	58	87	115	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	2709	17	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	4874	16	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	7321	15	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	4·0108	14	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	3315	13	53	107	160	213	267
77	4·3315	3662	4015	4373	4737	5107	5483	5864	6252	6646	7046	12					
78	4·7046	7453	7867	8288	8716	9152	9594	5·0045	5·0504	5·0970	5·1446	11					
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140	6713	10					
80	5·6713	7297	7894	8502	9124	9758	6·0405	6·1066	6·1742	6·2432	6·3138	9					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	7·0264	7·1154	8					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	8·0285	8·1443	7					
83	8·1443	2636	3863	5126	6427	7769	9152	9·0579	9·2052	9·3572	9·5144	6					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20	11·43	5					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95	14·30	4					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46	19·08	3					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27	28·64	2					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08	57·29	1					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0	∞	0					
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

Mean differences cease to be sufficiently accurate.

The cotangent of a small angle of *n* minutes of arc or the tangent of 90 minus *n* is very nearly equal to 3438 divided by *n*.

NATURAL COTANGENTS

LOGARITHMIC SINES

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences	
													1′ 2′ 3′	4′ 5′
0	−∞	7·2419	7·5429	7190	8439	9408	8·0200	8·0870	8·1450	8·1961	8·2419	89		
1	8·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206	5428	88		
2	8·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041	7188	87		
3	8·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326	8436	86	21 41 62	82 103
4	8·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	9403	85	16 32 48	64 80
5	8·9403	9489	9573	9655	9736	9816	9894	9970	9·0046	9·0120	9·0192	84	13 26 39	52 65
6	9·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	0859	83	11 22 33	44 55
7	9·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	1436	82	10 19 29	38 48
8	9·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	1943	81	8 17 25	34 42
9	9·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	2397	80	8 15 23	30 38
10	9·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	2806	79	7 14 20	27 34
11	9·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	3179	78	6 12 19	25 31
12	9·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	3521	77	6 11 17	23 28
13	9·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	3837	76	5 11 16	21 26
14	9·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	4130	75	5 10 15	20 24
15	9·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	4403	74	5 9 14	18 23
16	9·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4659	73	4 9 13	17 21
17	9·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4900	72	4 8 13	16 20
18	9·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	5126	71	4 8 11	15 19
19	9·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	5341	70	4 7 11	14 18
20	9·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	5543	69	3 7 10	14 17
21	9·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	5736	68	3 6 10	13 16
22	9·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	5919	67	3 6 9	12 15
23	9·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	6093	66	3 6 9	12 15
24	9·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	6259	65	3 6 8	11 14
25	9·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	6418	64	3 5 8	11 13
26	9·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	6570	63	3 5 8	10 13
27	9·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	6716	62	2 5 7	10 12
28	9·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	6856	61	2 5 7	9 12
29	9·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	6990	60	2 4 7	9 11
30	9·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	7118	59	2 4 6	9 11
31	9·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	7242	58	2 4 6	8 10
32	9·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	7361	57	2 4 6	8 10
33	9·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	7476	56	2 4 6	8 10
34	9·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	7586	55	2 4 6	7 9
35	9·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	7692	54	2 4 5	7 9
36	9·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	7795	53	2 3 5	7 9
37	9·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	7893	52	2 3 5	7 8
38	9·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	7989	51	2 3 5	6 8
39	9·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	8081	50	2 3 5	6 8
40	9·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	8169	49	1 3 4	6 7
41	9·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	8255	48	1 3 4	6 7
42	9·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	8338	47	1 3 4	6 7
43	9·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	8418	46	1 3 4	5 7
44	9·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	8495	45	1 3 4	5 6
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′ 2′ 3′	4′ 5′

LOGARITHMIC COSINES

LOGARITHMIC SINES

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences				
													1′	2′	3′	4′	5′
45	9·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	8569	44	1	2	4	5	6
46	9·8569	8677	8584	8591	8598	8606	8613	8620	8627	8634	8641	43	1	2	4	5	6
47	9·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	8711	42	1	2	3	5	6
48	9·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	8778	41	1	2	3	4	6
49	9·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	8843	40	1	2	3	4	5
50	9·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	8905	39	1	2	3	4	5
51	9·8905	8911	8917	8923	8929	8935	8041	8947	8953	8959	8965	38	1	2	3	4	5
52	9·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	9023	37	1	2	3	4	5
53	9·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	9080	36	1	2	3	4	5
54	9·9060	9085	9091	9096	9101	9107	9112	9118	9123	9128	9134	35	1	2	3	4	5
55	9·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	9186	34	1	2	3	3	4
56	9·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	9236	33	1	2	3	3	4
57	9·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	9284	32	1	2	2	3	4
58	9·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	9331	31	1	2	2	3	4
59	9·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	9375	30	1	1	2	3	4
60	9·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	9418	29	1	1	2	3	4
61	9·9418	9422	9427	9331	9435	9439	9443	9447	9451	9455	9459	28	1	1	2	3	3
62	9·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	9499	27	1	1	2	3	3
63	9·9499	9503	9507	2510	9514	9518	9522	9625	9629	9533	9537	26	1	1	2	3	3
64	9·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	9573	25	1	1	2	2	3
65	9·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	9607	24	1	1	2	2	3
66	9·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	9640	23	1	1	2	2	3
67	9·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	9672	22	1	1	2	2	3
68	9·9672	9676	9578	9681	9684	9687	9690	9693	9696	9699	9702	21	0	1	1	2	2
69	9·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	9730	20	0	1	1	2	2
70	9·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	9757	19	0	1	1	2	2
71	9·9757	9759	9762	9764	9767	7770	9772	9775	9777	9780	9782	18	0	1	1	2	2
72	9·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	9806	17	0	1	1	2	2
73	9·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	9828	16	0	1	1	2	2
74	9·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	9849	15	0	1	1	1	2
75	9·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	9869	14	0	1	1	1	2
76	9·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	9887	13	0	1	1	1	2
77	9·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	9904	12	0	1	1	1	1
78	9·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	9919	11	0	1	1	1	1
79	9·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	9934	10	0	0	1	1	1
80	9·9934	9935	9936	9937	9939	9940	9941	9942	9944	9945	9946	9	0	0	1	1	1
81	9·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	9958	8	0	0	1	1	1
82	9·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	9968	7	0	0	1	1	1
83	9·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	9976	6	0	0	0	1	1
84	9·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	9983	5	0	0	0	0	1
85	9·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	9989	4	0	0	0	0	0
86	9·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	9994	3	0	0	0	0	0
87	9·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	9997	2	0	0	0	0	0
88	9·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	9999	1	0	0	0	0	0
89	9·9999	9999	10·000	0000	0000	0000	0000	0000	0000	0000	0000	0					
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

LOGARITHMIC COSINES

LOGARITHMIC TANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences				
													1′	2′	3′	4′	5′
0	—∞	7·2419	7·5429	7·7190	7·8439	7·9409	8·0200	8·0870	8·1450	8·1962	8·2419	89					
1	8·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208	5431	88					
2	8·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046	7194	87					
3	8·7194	7337	7475	7609	7739	7865	7988	8107	8223	8336	8446	86					
4	8·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	9420	85	16	32	48	64	81
5	8·9420	9506	9591	9674	9756	9836	9915	9992	9·0068	9·0143	9·0216	84	13	26	40	53	66
6	9·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	0891	83	11	22	34	45	56
7	9·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	1478	82	10	20	29	39	49
8	9·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	1997	81	9	17	26	35	43
9	9·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	2463	80	8	16	23	31	39
10	9·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	2887	79	7	14	21	28	35
11	9·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	3275	78	6	13	19	26	32
12	9·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	3634	77	6	12	18	24	30
13	9·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	3968	76	6	11	17	22	28
14	9·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	4281	75	5	10	16	21	26
15	9·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	4575	74	5	10	15	20	25
16	9·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	4853	73	5	9	14	19	23
17	9·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	5118	72	4	9	13	18	22
18	9·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	5370	71	4	8	13	17	21
19	9·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	5611	70	4	8	12	16	20
20	9·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	5842	69	4	8	12	15	19
21	9·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	6064	68	4	7	11	15	19
22	9·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	6279	67	4	7	11	14	18
23	9·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	6486	66	3	7	10	14	17
24	9·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	6687	65	3	7	10	13	17
25	9·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	6882	64	3	7	10	13	16
26	9·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	7072	63	3	6	9	13	16
27	9·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	7257	62	3	6	9	12	15
28	9·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	7438	61	3	6	9	12	15
29	9·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	7614	60	3	6	9	12	15
30	9·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	7788	59	3	6	9	12	14
31	9·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	7958	58	3	6	9	11	14
32	9·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	8125	57	3	6	8	11	14
33	9·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	8290	56	3	5	8	11	14
34	9·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	8452	55	3	5	8	11	14
35	9·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	8613	54	3	5	8	11	13
36	9·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	8771	53	3	5	8	11	13
37	9·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	8928	52	3	5	8	10	13
38	9·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	9084	51	3	5	8	10	13
39	9·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	9238	50	3	5	8	10	13
40	9·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	9392	49	3	5	8	10	13
41	9·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	9544	48	3	5	8	10	13
42	9·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	9697	47	3	5	8	10	13
43	9·9697	9712	9727	9742	9757	9772	9788	9803	9818	9833	9848	46	3	5	8	10	13
44	9·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	1000	45	3	5	8	10	13
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

LOGARITHMIC COTANGENTS

LOGARITHMIC TANGENTS

Degrees	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Degrees	Mean Differences				
													1′	2′	3′	4′	5′
45	10·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	0152	44	3	5	8	10	13
46	10·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	0303	43	3	5	8	10	13
47	10·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	0456	42	3	5	8	10	13
48	10·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	0608	41	3	5	8	10	13
49	10·0608	0624	0639	0654	0670	0685	0700	0715	0731	0746	0762	40	3	5	8	10	13
50	10·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	0916	39	3	5	8	10	13
51	10·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	1072	38	3	5	8	10	13
52	10·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	1229	37	3	5	8	10	13
53	10·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	1387	36	3	5	8	11	13
54	10·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	1548	35	3	5	8	11	13
55	10·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	1710	34	3	5	8	11	14
56	10·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	1875	33	3	5	8	11	14
57	10·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	2042	32	3	6	8	11	14
58	10·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	2212	31	3	6	9	11	14
59	10·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	2386	30	3	6	9	12	14
60	10·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	2562	29	3	6	9	12	15
61	10·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	2743	28	3	6	9	12	15
62	10·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	2928	27	3	6	9	12	15
63	10·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3118	26	3	6	9	13	16
64	10·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3313	25	3	6	10	13	16
65	10·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3514	24	3	7	10	13	17
66	10·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3721	23	3	7	10	14	17
67	10·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	3936	22	4	7	11	14	18
68	10·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4158	21	4	7	11	15	19
69	10·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4389	20	4	8	12	15	19
70	10·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4630	19	4	8	12	16	20
71	10·4630	4655	4680	4705	4730	4755	4780	4806	4831	4857	4882	18	4	8	13	17	21
72	10·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	5147	17	4	9	13	18	22
73	10·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5425	16	5	9	14	19	23
74	10·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5719	15	5	10	15	20	25
75	10·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	6032	14	5	10	16	21	26
76	10·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6366	13	6	11	17	22	28
77	10·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6725	12	6	12	18	24	30
78	10·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	7113	11	6	13	19	26	32
79	10·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7537	10	7	14	21	28	35
80	10·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8003	9	8	16	23	31	39
81	10·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	8522	8	9	17	26	35	43
82	10·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	9109	7	10	20	29	39	49
83	10·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	9784	6	11	22	34	45	56
84	10·9784	9857	9932	11·0008	11·0085	11·0164	11·0244	11·0326	11·0409	11·0494	11·0580	5	13	26	40	53	66
85	11·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	1554	4	16	32	48	64	81
86	11·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663	2806	3					
87	11·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357	4569	2					
88	11·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167	7581	1	Differences				
89	11·7581	8038	8550	9130	9800	12·0591	12·1561	12·2810	12·4571	12·7581	+∞	0	vary rapidly.				
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′		1′	2′	3′	4′	5′

LOGARITHMIC COTANGENTS

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The Quaid Said.

"You must devote yourself whole-heartedly to your studies, for that is your first obligation to yourselves, your parents and to the state."

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It is Injurious to Health

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